



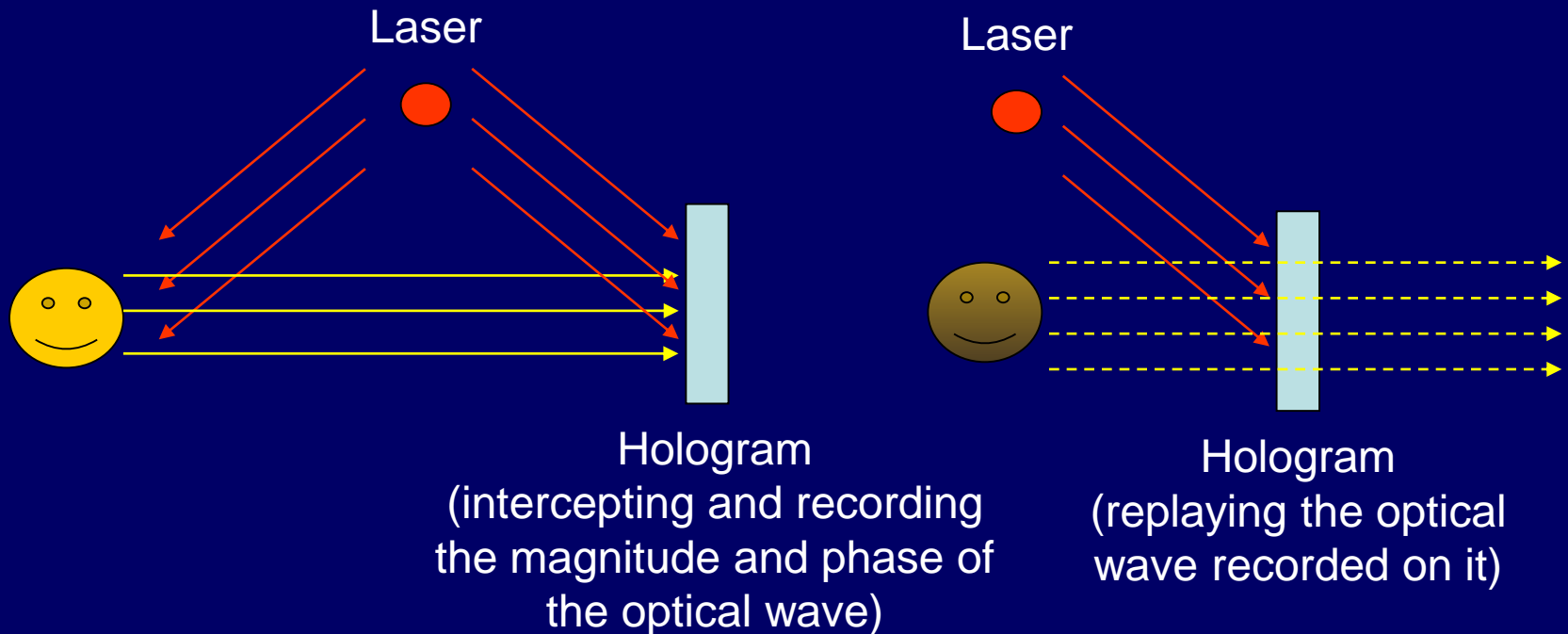
Computer Generated Holograms

Dr. P.W.M. Tsang



Optical Generated Holography

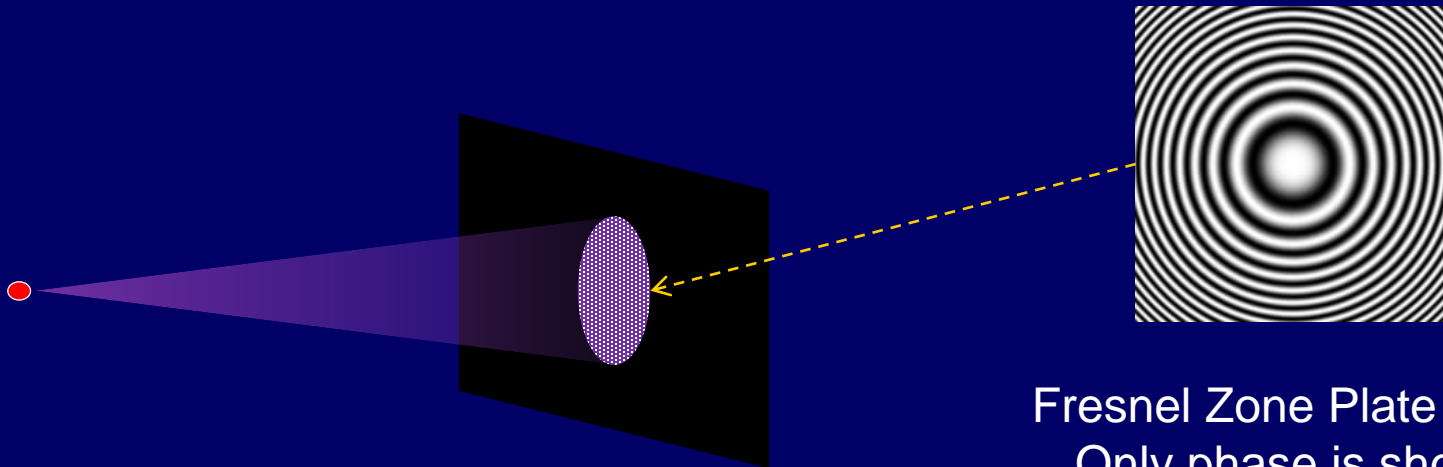
Hologram: Recording and reproduction of 3D scene on and from a 2D media (such as film).





Optical Generated Holography

What is the wavefront looks like on the hologram? Consider a single object point.

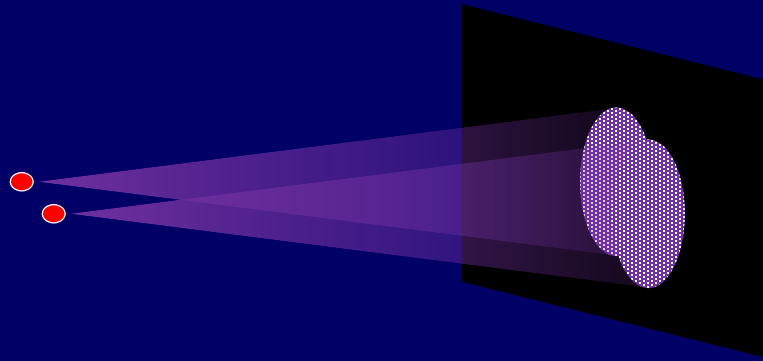


Fresnel Zone Plate (FZP)
Only phase is shown.
Magnitude is constant.



Optical Generated Holography

What about multiple object point: superposition theory



For example, 2 object points,
FZPs added together on the
hologram



Optical Generated Holography

Mathematical expression of a FZP

$$FZP(x, y; z_{m;n}) = \exp \left\{ -j2\pi\lambda^{-1} \sqrt{x^2 + y^2 + z_{m;n}^2} \right\}$$

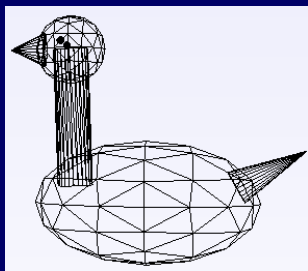
Light from a point source spread in all directions. When intercept by an opaque media, the optical signal will be in the form of a constant magnitude function known as a Fresnel Zone Plate (FZP).

When more than one point sources are present, individual FZPs will sum up on the opaque media that intercepts the optical waves. A digital hologram can be computed on this basis.



Computer Generated Holography

Computer Generated Hologram (CGH): Generation of holograms numerically from three dimensional (3-D) models that do not actually exist in the real world.



Computer

Hologram file

Printer/
Display

hologram

3D Computer graphic model

Given a discrete, 3-D image, a Fresnel hologram can be generated numerically as the real part of the product of the object and a planar reference waves. **The 3-D image can be reconstructed from the hologram afterwards.**



Computer Generated Holography: Fresnel Hologram

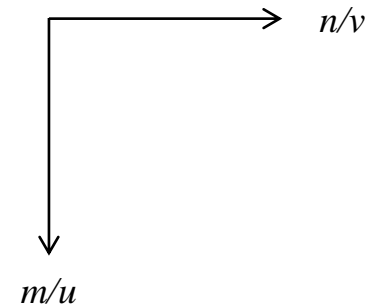
Given a three dimensional (3D) surface with an intensity distribution $I(m,n)$, the Fresnel hologram is given by

$$O(u, v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} I(m, n) \exp \left\{ -j2\pi\lambda^{-1} \underbrace{\sqrt{[(m-u)p]^2 + [(n-v)p]^2 + z_{m;n}^2}} \right\}$$

p is the pixel size

Distance of a point at (m,n) to a point at (u,v) on the hologram

$z_{m;n}$ is the perpendicular distance of the object point to the hologram





Computer Generated Holography: Fresnel Hologram

Given a three dimensional (3D) surface with an intensity distribution $I(m,n)$, the Fresnel hologram is given by

$$O(u, v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} I(m, n) \exp \left\{ -j2\pi\lambda^{-1} \sqrt{[(m-u)p]^2 + [(n-v)p]^2 + z_{m,n}^2} \right\}$$

Table 1 Arithmetic calculations of a single operation for computing the hologram

Arithmetic calculation	Number
Multiplication	2
Square	3
Square root	1
Exponential function	1

Very heavy computation.



Computer Generated Holography: Fresnel Hologram

Given a three dimensional (3D) surface with an intensity distribution $I(m,n)$, the Fresnel hologram is given by the convolution of $I(m,n)$ with the FZP

$$O(u, v) = I(u, v) * FZP(u, v)$$

Convolution is tedious, a better way is to conduct it in the frequency space

$$O(\omega_u, \omega_v) = I(\omega_u, \omega_v) \times FZP(\omega_u, \omega_v)$$

With FFT, fourier transform can be performed swiftly. The hologram can be generated with point to point multiplication, which is more computation efficient. However, the above is only for a single plane. The computation will become more heavy with increasing image planes.

If the hologram is complex, the object scene can be fully reconstructed numerically

$$I(\omega_u, \omega_v) = \frac{O(\omega_u, \omega_v)}{FZP(\omega_u, \omega_v)}$$



Computer Generated Holography: Fast algorithm

$$A \exp \left\{ -j2\pi\lambda^{-1} \sqrt{[(m-u)p]^2 + [(n-v)p]^2 + z_{m;n}^2} \right\}$$

Precompute the result of the above equation for all combinations of the 6 variables (A, m, n, u, v, z) .

The memory is known as a look up table (LUT). Each cell in the LUT can be retrieved by specifying the 6 variables as indices. Computation of the hologram is reduced to memory look-up and simple addition.

$$O(u, v) = \sum_m \sum_n L(I(m, n), m, n, u, v, z_{m;n})$$

However the memory required is extremely huge even for modern computers.



Computer Generated Holography: Novel LUT (N-LUT)

$$FZP(m, n; z_{m;n}) = \exp \left\{ -j2\pi\lambda^{-1} \sqrt{m^2 + n^2 + z_{m;n}^2} \right\}$$

We can infer that

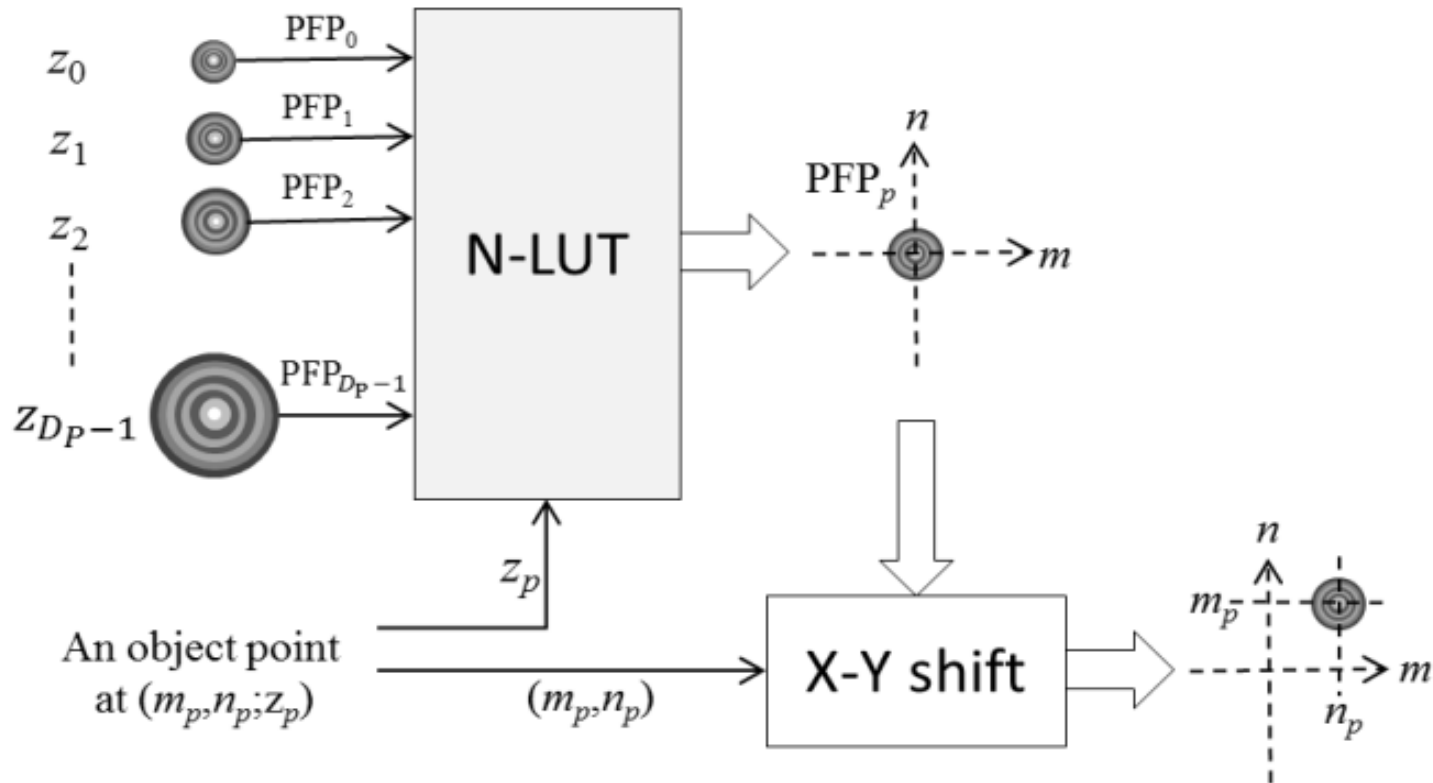
$$\exp \left\{ -j2\pi\lambda^{-1} \sqrt{[(m-u)p]^2 + [(n-v)p]^2 + z_{m;n}^2} \right\} = FZP(m-u, n-v, z_{m;n})$$

The LUT can be reduced to one that is dependent on 3 variables: m , n , and $z_{m;n}$. In the LUT the values of the function $FZP(m, n; z_{m;n})$ (which is known as the principal fringe pattern or the N-LUT) for all combinations of the 3 variables are stored. The hologram can be obtained as

$$O(u, v) = \sum_m \sum_n I(m, n) FZP(m-u, n-v; z_{m;n})$$



Computer Generated Holography: Novel LUT (N-LUT)



The N-LUT method



Computer Generated Holography: Novel LUT (N-LUT)

Memory size of LUT and N-LUT

- Hologram/image size = 512x512
- Intensity quantization: 256 levels.
- Number of depth planes (z) = 16
- Number of bits of each LUT entry=1 byte

LUT: $256 \times 512 \times 512 \times 512 \times 512 \times 16 = 281478\text{Gbytes}$

N-LUT: $512 \times 512 \times 16 = 4.2\text{Mbytes}$

The N-LUT is much smaller in size than the LUT, but a bit more calculations (multiplying intensity with the FZP, and translating the PFP vertically and horizontally) are required in generating the hologram.

$$O(u, v) = \sum_m \sum_n I(m, n) FZP(m - u, n - v; z_{m;n})$$



Computer Generated Holography: Split LUT (S-LUT)

Consider the optical wave of a point source at location (m,n) , falling on a point (u,v) on the hologram. Axial distance between point and hologram = $z_{m;n}$.

$$\exp \left\{ -j2\pi\lambda^{-1} \sqrt{[(m-u)p]^2 + [(n-v)p]^2 + z_{m;n}^2} \right\}$$

Rewriting the equation, we have

$$\exp \left\{ -j2\pi\lambda^{-1} \sqrt{\Delta_m^2 + \Delta_n^2 + z_{m;n}^2} \right\}, \text{ where}$$

$$\Delta_m = |m-u|p, \Delta_n = |n-v|p.$$



Computer Generated Holography: Split LUT (S-LUT)

Assuming $\Delta_m \ll z_p$, $\Delta_n \ll z_p$, and z_p is integer multiple of λ , and let $w_n = \frac{2\pi}{\lambda}$, the above expression can be approximated as

$$\exp\left\{-j2\pi\lambda^{-1}\sqrt{\Delta_m^2 + \Delta_n^2 + z_{m;n}^2}\right\} = \exp[iw_n(\Delta_m^2 + z_{m;n}^2)]\exp[iw_n(\Delta_n^2 + z_{m;n}^2)]$$
$$= O_H(\Delta_m, z_{m;n})O_V(\Delta_n, z_{m;n}).$$

$O_H(\Delta_m, z_{m;n})$, $O_V(\Delta_n, z_{m;n})$ are known as the horizontal and the vertical light modulators.

A small LUT (known as S-LUT) will be sufficient to store all combinations of the light modulators.



Computer Generated Holography: Split LUT (S-LUT)

Memory size of N-LUT and S-LUT

- Hologram/image size = 512×512 (Δ_m or Δ_n restricted to 512)
- Number of depth planes (z) = 16
- Number of bits of each LUT entry = 1 byte

N-LUT: $512 \times 512 \times 16 = 4.2\text{Mbytes}$

S-LUT: $512 \times 16 = 8.2\text{Kbytes}$

The S-LUT is much smaller in size than the N-LUT, but a bit more calculations (multiplying intensity with the pair of light modulators, and computing Δ_m and Δ_n) are required in generating the hologram.

$$O(u, v) = \sum_m \sum_n I(m, n) O_H(\Delta_m, z_{m;n}) O_V(\Delta_n, z_{m;n})$$



Computer Generated Holography: LUT, N-LUT and S-LUT)



Decreasing memory size of LUT: significant

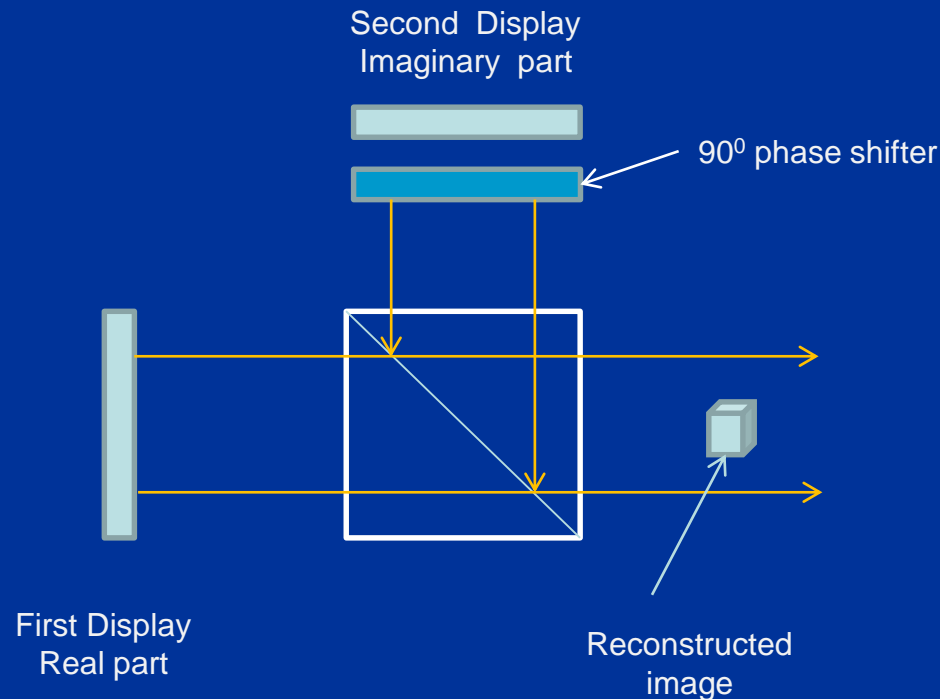
Increasing amount of computation: minor

LUT approach does not simplify the hologram formation process.



Displaying Digital Fresnel Hologram

Displaying a complex hologram optically using 2 Amplitude Spatial light modulators (SLMs)

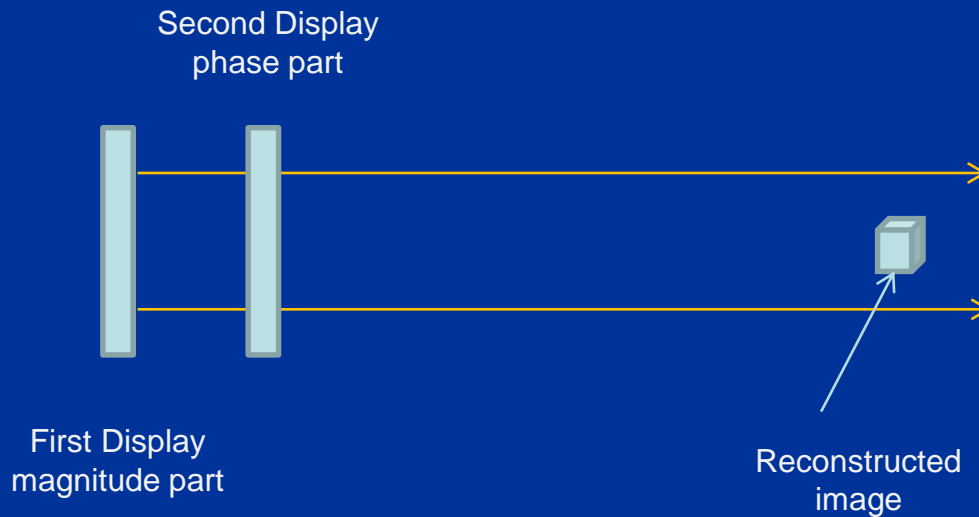


Both displays are amplitude only SLM



Displaying Digital Fresnel Hologram

Displaying a complex hologram optically? An Amplitude and a phase Spatial light modulator

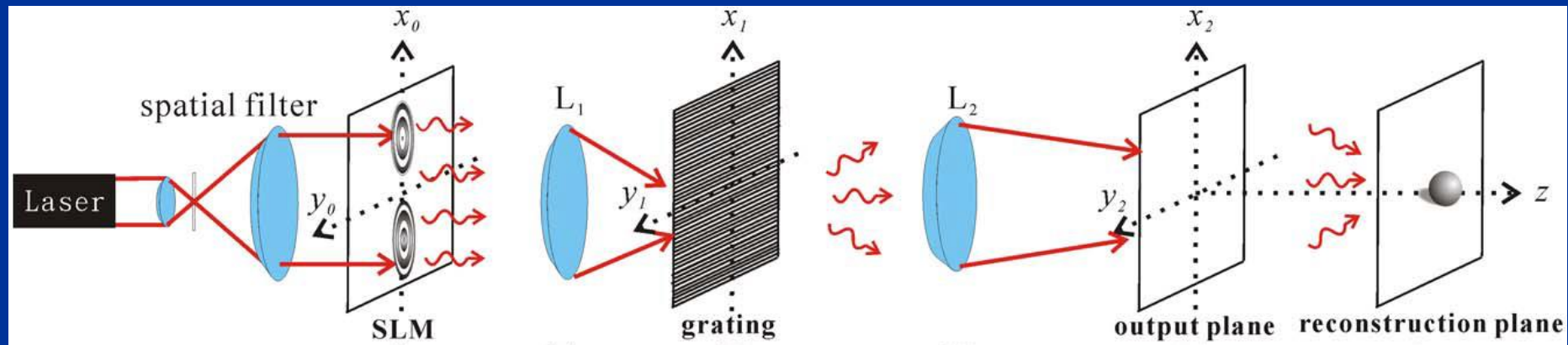


Cascading an amplitude only and a phase only SLMs



Displaying Digital Fresnel Hologram

Displaying a complex hologram optically with an amplitude-only SLM and a high resolution grating



Excerpted from J. Liu, W. Hsieh, T. Poon, and P. Tsang, "Complex Fresnel hologram display using a single SLM," *Appl. Opt.* 50, H128-H135 (2011).

- Real and Imaginary holograms displayed at different vertical sections on the SLM
- The lens perform the Fourier Transform
- The sinusoidal grating couples the real and the imaginary components on the Fourier Plane
- The signal at the output of the grating is Fourier Transform to deliver the reconstructed image



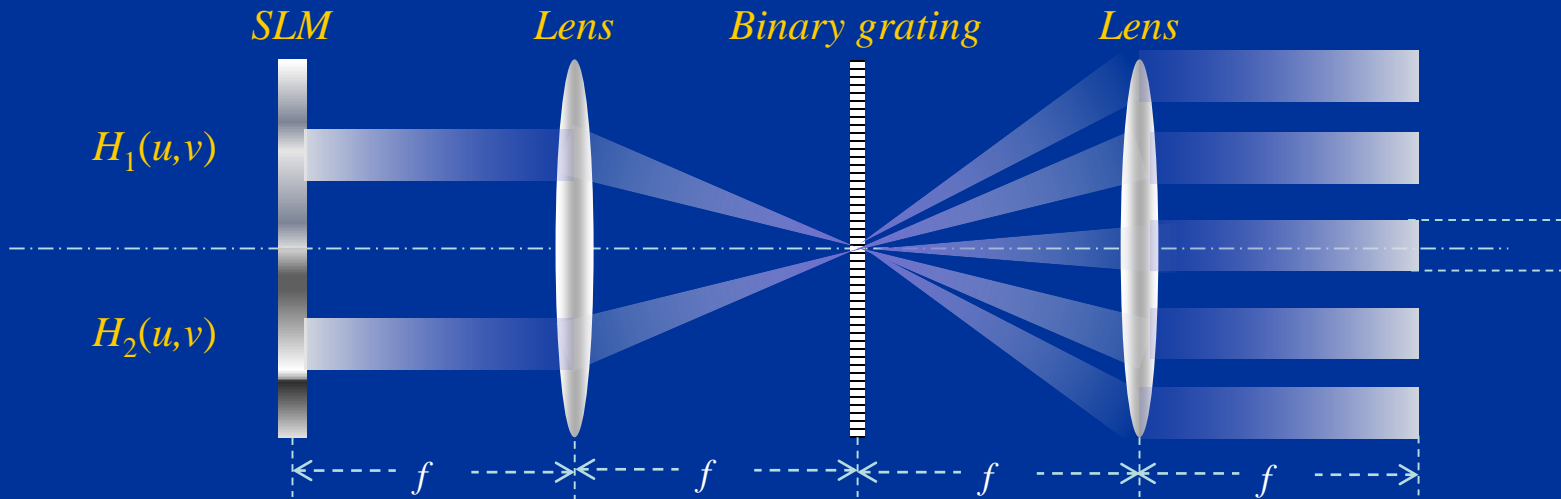
Displaying Digital Fresnel Hologram

Displaying a complex hologram optically with a phase-only SLM, lens and binary grating.

Any complex number can be converted into the sum of a pair of phase-only quantities.

$$H(u, v) = \exp^{i\theta_1(u, v)} + \exp^{i\theta_2(u, v)} = H_1(u, v) + H_2(u, v)$$

C. Hsueh and A. Sawchuk, "Computer-generated double-phase holograms," Appl. Opt. 17, 3874-3883 (1978).



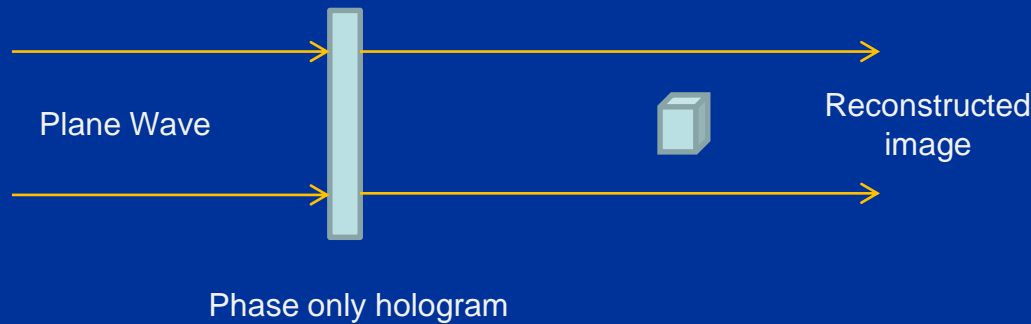
H. Song, G. Sung, S. Choi, K. Won, H. Lee, and H. Kim, "Optimal synthesis of double-phase computer generated holograms using a phase-only spatial light modulator with grating filter," Opt. Express 20, 29844-29853 (2012).



Displaying Digital Fresnel Hologram

Displaying a complex hologram optically in phase-only SLM without lens

Set the magnitude of the complex hologram to a constant value, while the phase remains intact.



Disadvantage:
heavy
distortion on
the
reconstructed
image



Reconstructed image of a complex hologram



Reconstructed image of the phase component of a complex hologram



Displaying Digital Fresnel Hologram

Displaying a complex hologram optically in phase-only SLM without lens



A 40+ years problem,
but why still an area
of immense interest?



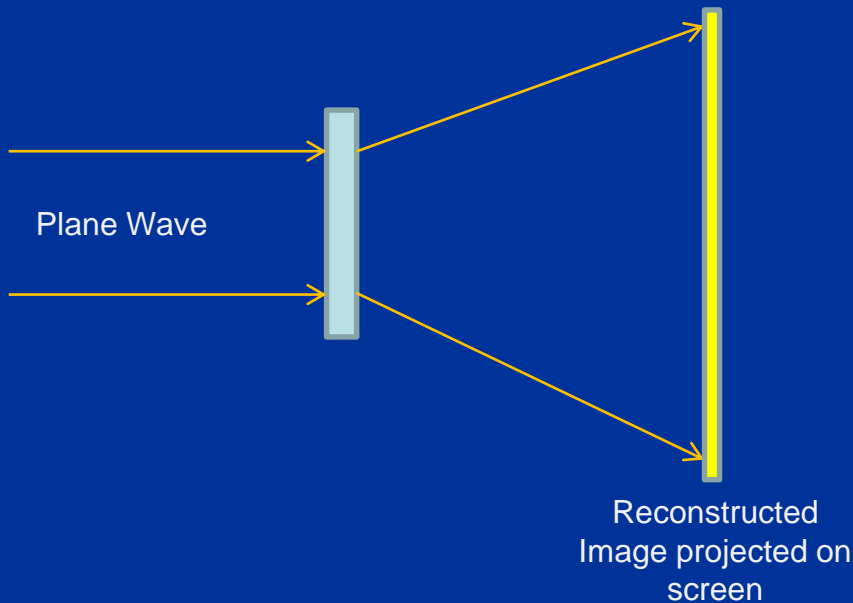
Reconstructed image of the
phase component of a complex
hologram



Displaying Digital Fresnel Hologram

Len free holographic projection system:
Electronic focusing

Enormous Market Potential



- Higher optical efficiency compares with amplitude holograms
- Free from twin images and zero order diffraction
- Easy to set focal plane, hence suitable for lens free holographic projection



Displaying Digital Fresnel Hologram

Complex modulation

$$\exp\left\{s\left(t - t^{-1}\right)/2\right\} = \sum_{m=-\infty}^{\infty} t^m J_m(s)$$

$$\exp[is \cos \phi] = \sum_{m=-\infty}^{\infty} i^m J_m(s) \exp(im\phi)$$

$J_m(s)$ Bessel function.

Let $t = \sqrt{-1} = i$, we have

$$c \exp\left[s\left(i - i^{-1}\right)/2\right] = c \exp[is] = c \sum_{m=-\infty}^{\infty} i^m J_m(s)$$

Target hologram to be displayed

$$H(x, y) = |H(x, y)| \exp(i\theta(x, y))$$

Generate a phase hologram instead

$$H_P(x, y) = c \{i\beta |H(x, y)| \cos[\theta(x, y) - \theta_R(x, y)]\}$$

After mixing with the reference beam $\exp(i\theta_R)$

$$D_P(x, y) = c \exp(i\theta_R) \{i\beta |H(x, y)| \cos[\theta(x, y) - \theta_R(x, y)]\}$$

$$= c \sum_{-\infty}^{\infty} J_m[\beta |H(x, y)|] i^m \exp\{-i[m\theta(x, y) - (m+1)\theta_R]\}$$

Different values of m diffracts the reconstructed beam at different angles.

When $m=-1$, we have

$$D_P(x, y) = c \sum_{-\infty}^{\infty} J_{-1}[\beta |H(x, y)|] i^{-1} \exp\{i[\theta(x, y)]\}$$



Displaying Digital Fresnel Hologram

Complex modulation

Target hologram to be displayed

$$H(x, y) = |H(x, y)| \exp(i\theta(x, y))$$

Generate a phase hologram instead

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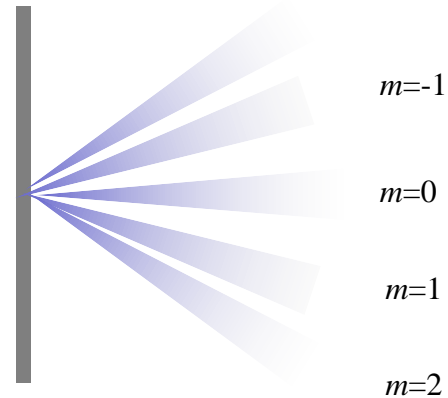
$$= c \sum_{-\infty}^{\infty} J_m[\beta |H(x, y)|] i^m \exp\{-i[m\theta(x, y) - (m+1)\theta_R]\}$$

Different values of m diffracts the reconstructed beam at different angles.

When $m=-1$, we have

$$D_P(x, y) = c \sum_{-\infty}^{\infty} J_{-1}[\beta |H(x, y)|] i^{-1} \exp\{i[\theta(x, y)]\}$$

Phase hologram

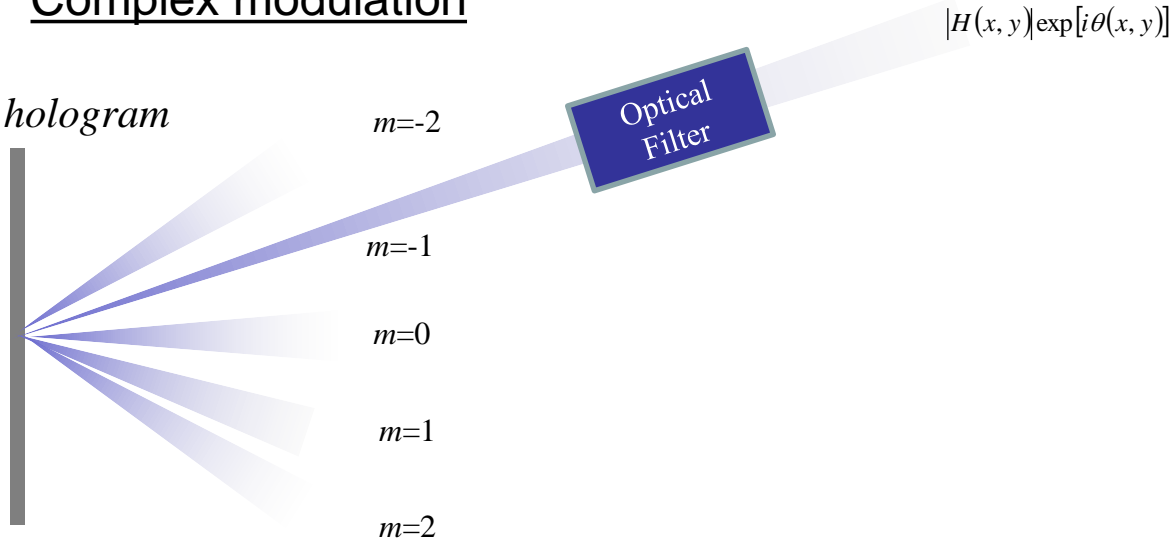




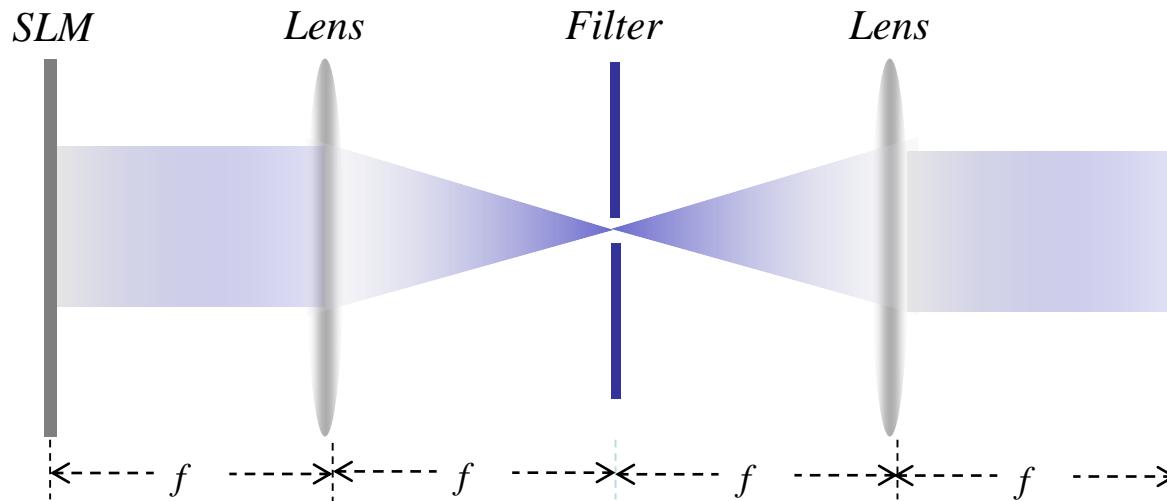
Displaying Digital Fresnel Hologram

Complex modulation

Phase hologram



Optical filter



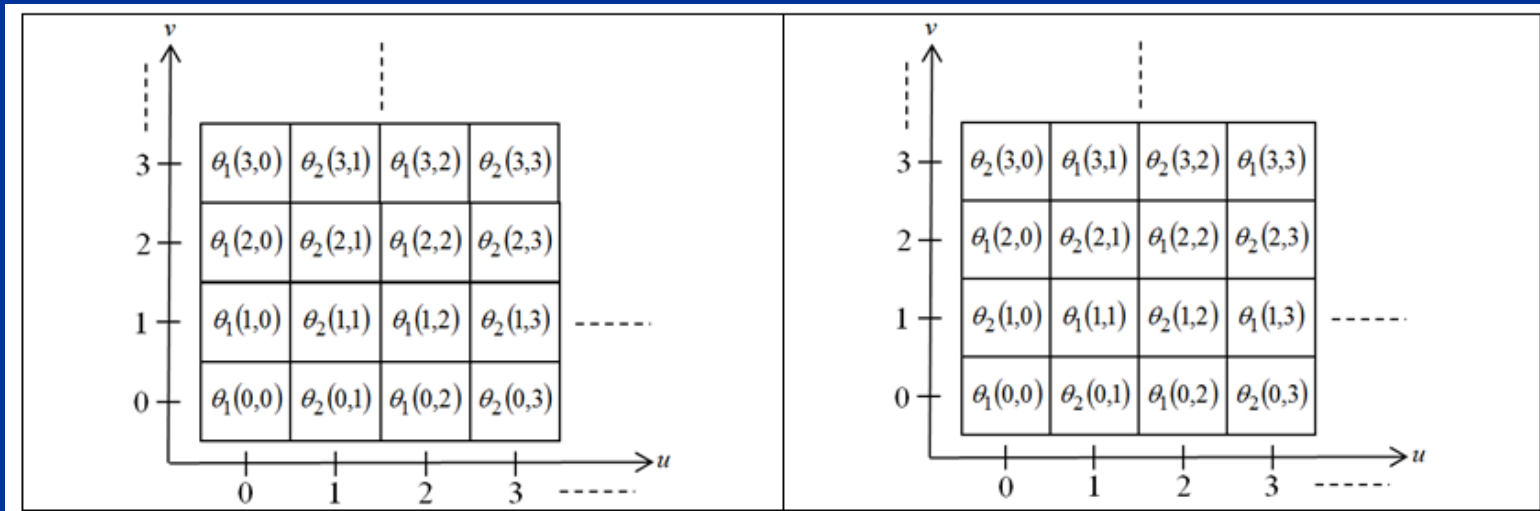


Double Phase Macro Pixel Hologram

Displaying a complex hologram optically in phase-only SLM without lens: Macropixel

If resolution of SLM is high enough, spatially multiplex the pair of phase components in a uniform manner

$$H(u, v) = 0.5 \left[\exp^{i\theta_1(u,v)} + \exp^{i\theta_2(u,v)} \right]$$



1 × 2

2 × 2



Double Phase Macro Pixel Hologram

$$H(u, v) = 0.5 \left[\exp^{i\theta_1(u, v)} + \exp^{i\theta_2(u, v)} \right]$$

It can be proved that the magnitude and phase components of the hologram can be derived from the pair of phase angles θ_1 and θ_2 .

Spatial division multiplexing of the pair of phase components is a downsampling process that can lead to aliasing error.

Different spatial division multiplexing of the pair of phase components can lead to different quality of the reconstructed images. Lets have a look at 2 popular multiplexing topologies, the 1×2 and the 2×2 macro pixel format.



Double Phase Macro Pixel Hologram

Evaluation on reconstructed images (intensity amplified by around 10 times).



1×2

2×2

The method is fast and the visual quality is good.

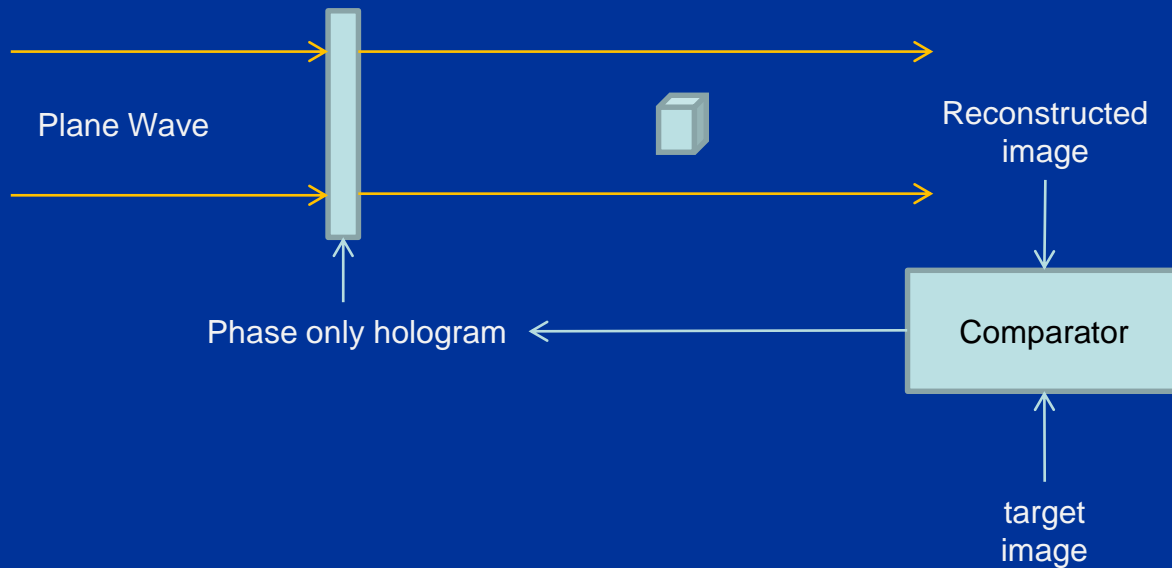
Noise is prominent in the 1×2 structure, and less in the 2×2 structure.

The intensity is low.



Displaying Digital Fresnel Hologram

Converting complex hologram to phase only image using the iterative approach



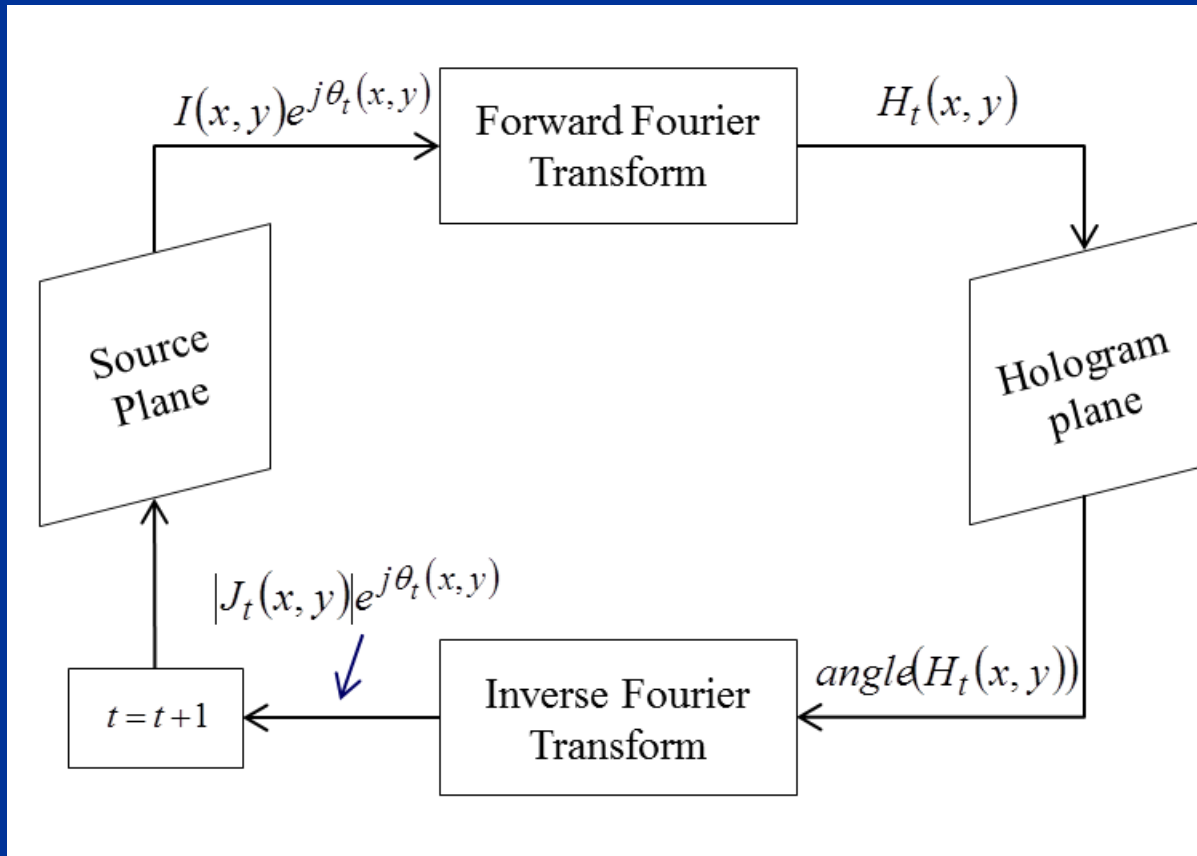
Disadvantage:
heavy amount of
computation in the
iterative process,
especially if multiple
depth images is
involved.

Adjust the phase only hologram until the reconstructed image is same as the target ones.



Gerchberg Saxton algorithm (GSA)

Generating phase-only Fourier hologram from an image using the iterative approach, based on principles of GSA.



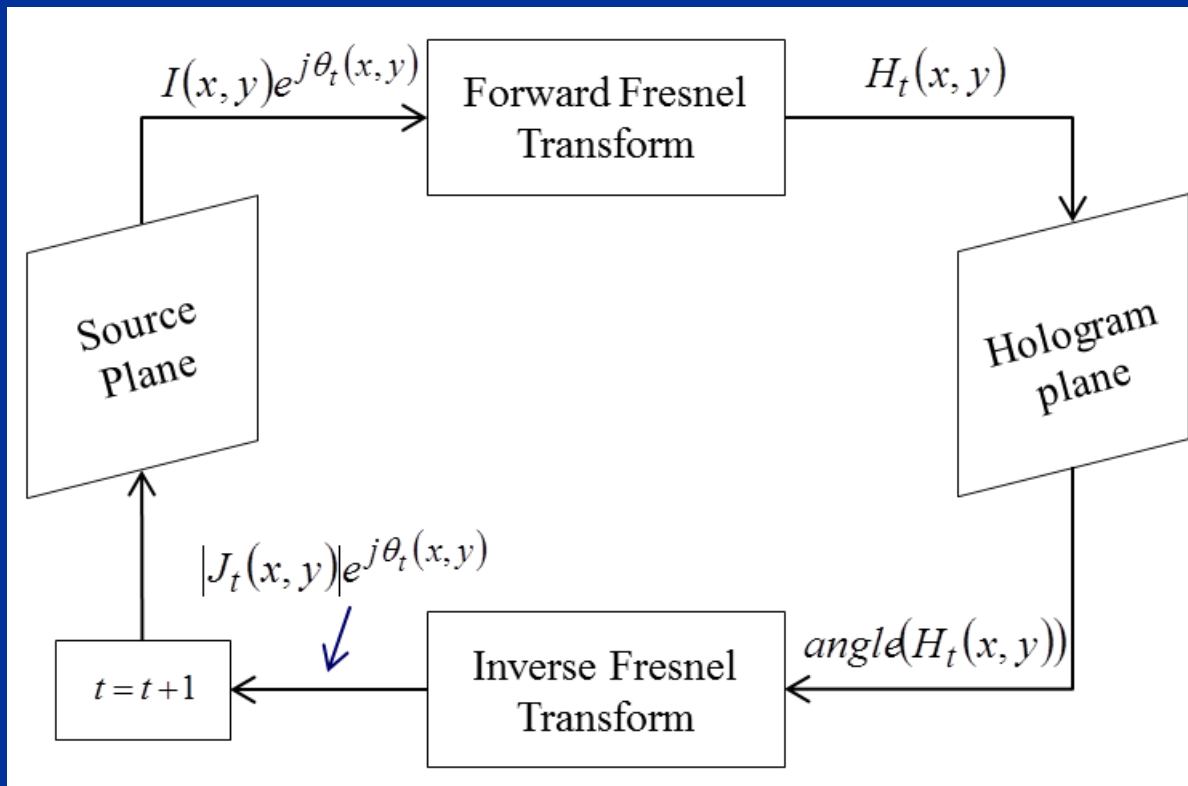
1. Given an image $I(x, y)$, to be converted to a hologram.
2. Generated the Fourier hologram $H(x, y)$ for $I(x, y)$.
3. Keep the phase component, and revert back to the spatial image with IFT,
4. Get the image and the phase of the inverse transformed hologram.
5. Repeat 2 to 4 until the error is smaller than a threshold.

iterative Fourier transform algorithm (IFTA).



Gerchberg Saxton algorithm (GSA)

Generating phase-only Fresnel hologram from an image using the iterative Fresnel transform algorithm (IFTA).

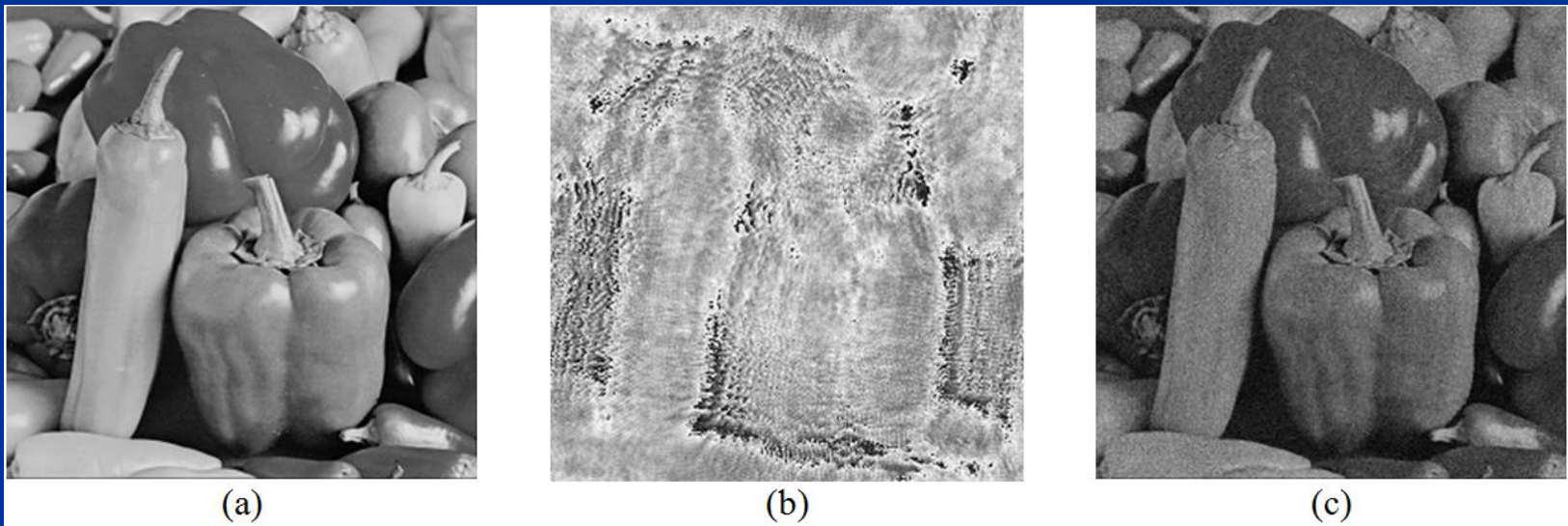


1. Given an image $I(x, y)$, to be converted to a hologram.
2. Generated the Fresnel hologram $H(x, y)$ for $I(x, y)$.
3. Keep the phase component, and revert back to the spatial image with inverse Fresnel transform,
4. Get the image and the phase of the inverse Fresnel transformed hologram.
5. Repeat 2 to 4 until the error is smaller than a threshold.



Gerchberg Saxton algorithm (GSA)

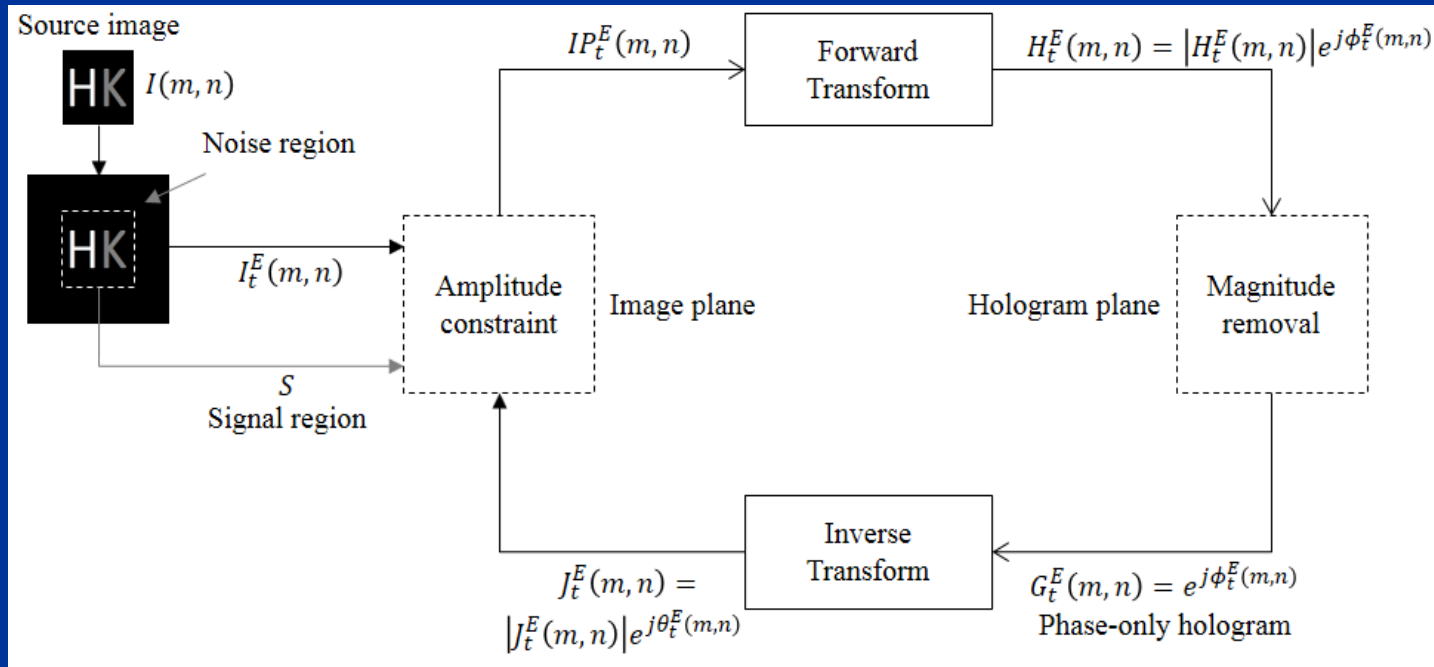
Generating phase-only Fresnel hologram from an image using the iterative Fresnel transform algorithm (IFTA).



(a) Source image "Peppers", (b) Phase-only hologram of the image "Peppers", obtained with the GSA, (c) Reconstructed image of the phase-only hologram in (b).



Mixed-region Amplitude Freedom (MRAF)



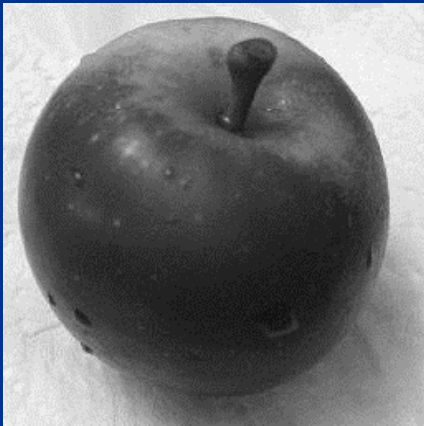
1. Source image is divided into a signal and a noise region.
2. For the signal region, amplitude constraint is imposed.
3. For the noise region, there is no amplitude constraint.

$$IP_t^E(m,n) = \begin{cases} I^E(m,n) & \text{if } (m,n) \in S \\ J_t^E(m,n) & \text{otherwise} \end{cases}$$

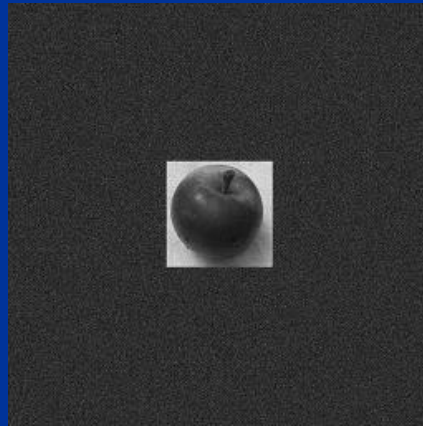
Noise region provides additional freedom to absorb the error in the signal region



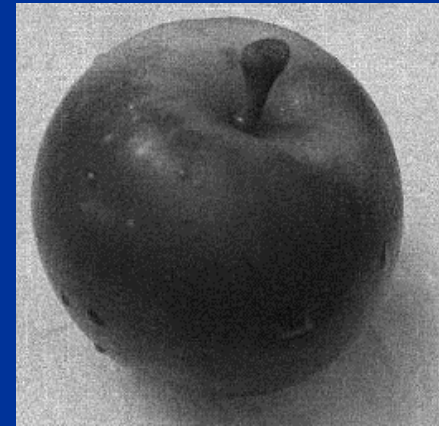
Mixed-region Amplitude Freedom (MRAF)



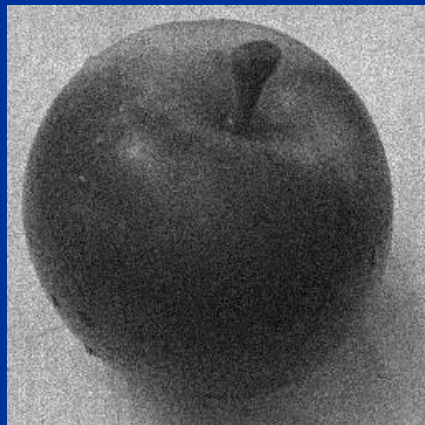
(a) Source image "Apple"



(b) Reconstructed image of phase-only hologram obtained with 5 rounds of MRAF



(c) Signal region of reconstructed image



(d) Reconstructed image of phase-only hologram obtained with 5 rounds of IFTA



Random noise addition (RNA)

Simulate the effect of overlaying an optical diffuser onto the image.

The diffuser scatters the optical waves so that its magnitude distribution is roughly homogeneous on the hologram. The phase component alone, therefore, is sufficient to represent the hologram.

$$I_N(m, n) = I(m, n) \times \aleph(m, n) = I(m, n) \times \exp[i\theta(m, n)].$$

$\theta(m, n)$ is a 2-D array of random values in the range $[0, 2\pi)$.

To generate a phase-only hologram, the image is first added without random phase noise, and converted into a Fresnel hologram. The magnitude of the hologram is set to unity, resulting in a phase-only hologram.

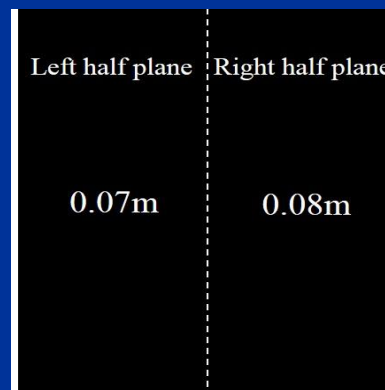
However, the reconstructed image is contaminated with noise.



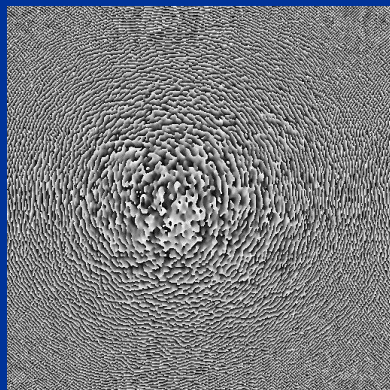
Random noise addition (RNA)



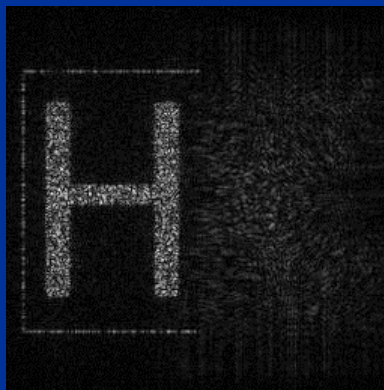
(a) Intensity distribution of a double-depth image.



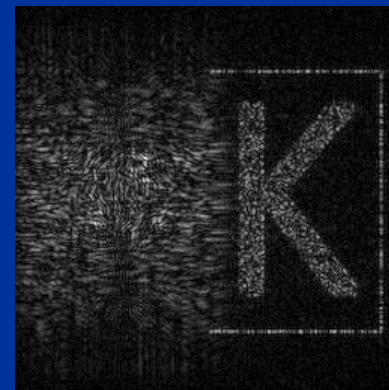
(b) Depth map of the double-depth image.



(c) Phase-only hologram obtained with RNA.



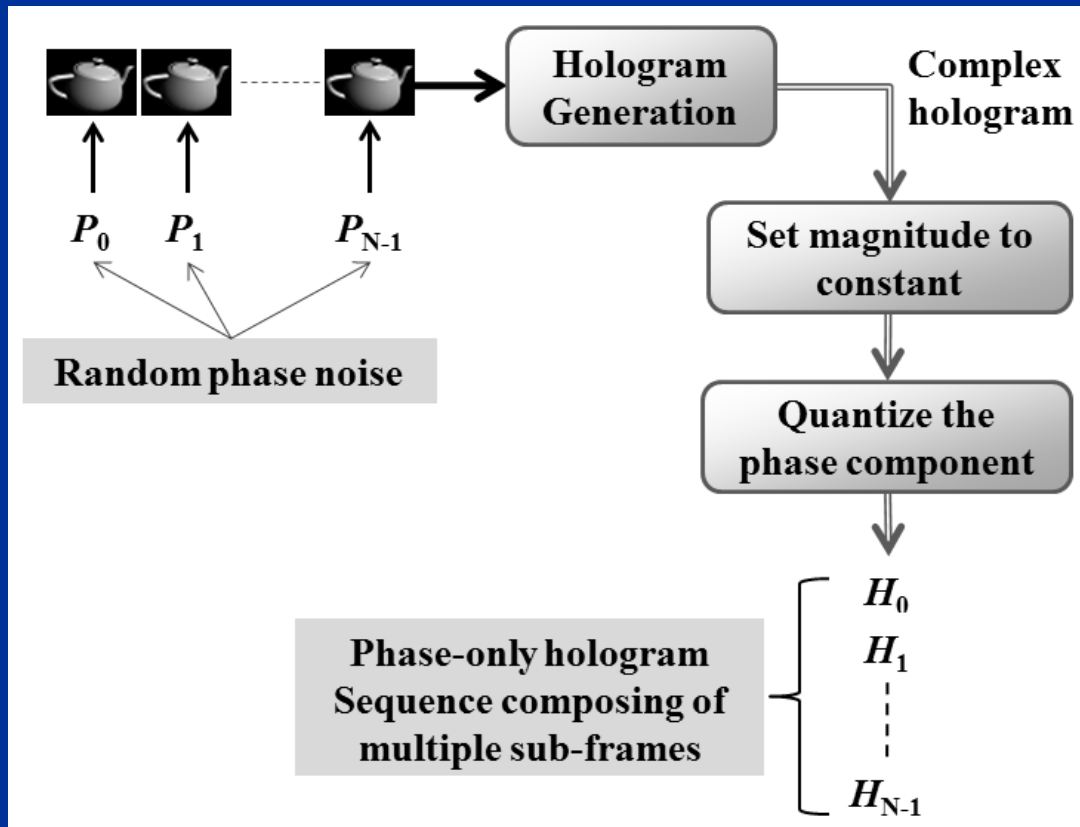
(d) Reconstructed image on first depth plane.



(e) Reconstructed image on second depth plane.



One Step Phase Retrieval Phase Only Hologram



Reconstructed images of hologram sub-frames are displayed sequentially at high frame rate. The noise is smoothed out with persistence of vision of human eyes



Multiple Sub-frames One Step Phase Retrieval



(a) (b) and (c): Simulated reconstructed image of a single phase-only hologram of the source image "Lenna", generated by the OSPR method, based on 1, 5, and 15 phase-only hologram(s), respectively.



One Step Phase Retrieval Phase Only Hologram

Advantages: Faster than iterative methods, and favorable visual quality on the reconstructed images.

Disadvantages

Multiple frames are required, and noise may not average out completely.
Restricted to object scene with specific characteristics (e.g. diffusive).

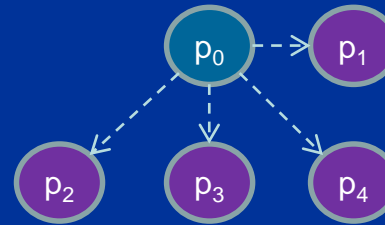
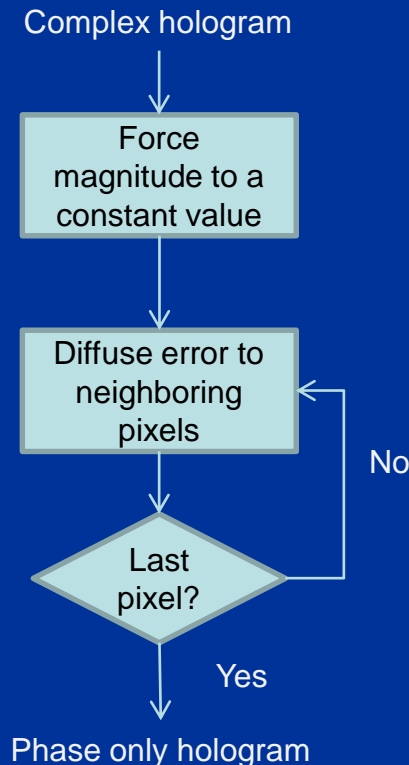
Very high frame rate is required, increasing the requirement and cost of the display device.

Intensive computation required to generate multiple frame holograms for a given object scene, especially for large hologram size.



Uni-directional Error Diffusion (UERD) Phase Only Hologram

- Scan each row of the complex hologram from left to right.
- Forced the magnitude of each scanned pixel to unity
- Diffuse error to the neighborhood, unvisited pixels (Floyd-Steinberg error diffusion)

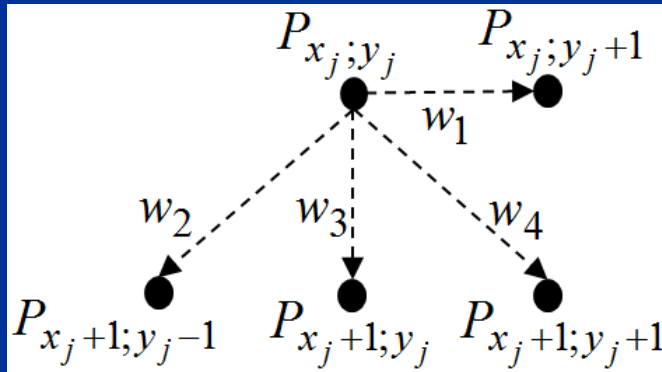


Advantage: Low complexity and high reconstructed image quality

P. Tsang and T. Poon, "Novel method for converting digital Fresnel hologram to phase-only hologram based on bidirectional error diffusion," Opt. Express 21, 23680-23686 (2013).



Uni-directional Error Diffusion (UERD) Phase Only Hologram



$$\begin{aligned}w_1 &= 7/16 \\w_2 &= 3/16 \\w_3 &= 5/16 \\w_4 &= 1/16\end{aligned}$$

Error

$$H(x_j, y_j + 1) \leftarrow H(x_j, y_j + 1) + w_1 E(x_j, y_j)$$

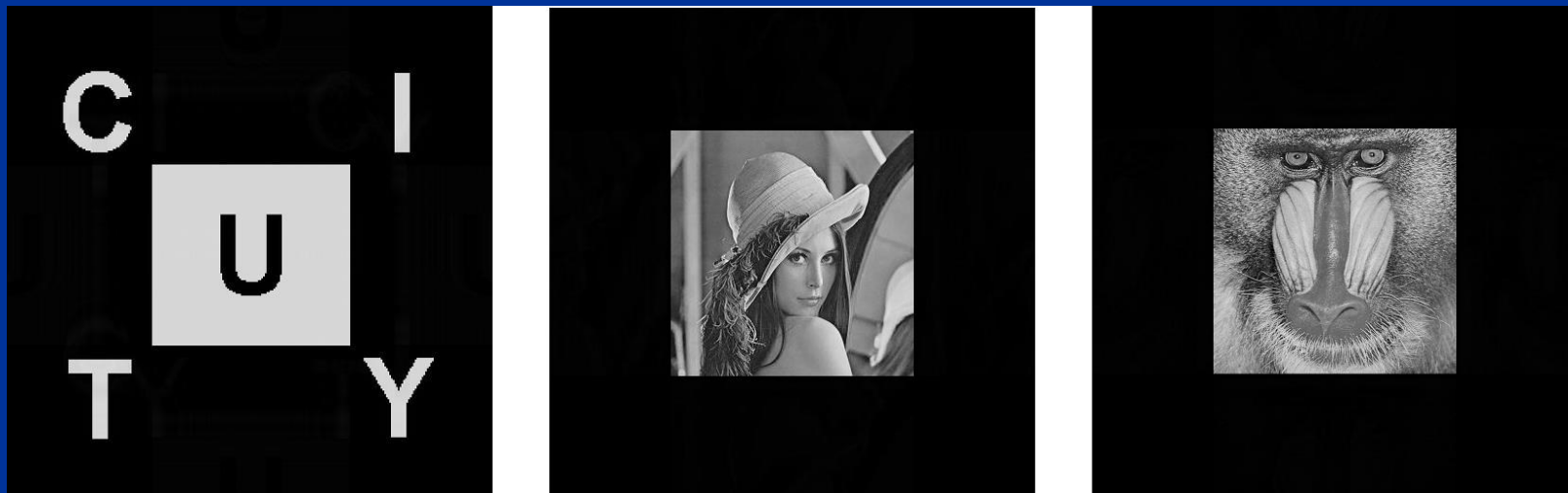
$$H(x_j + 1, y_j - 1) \leftarrow H(x_j + 1, y_j - 1) + w_2 E(x_j, y_j)$$

$$H(x_j + 1, y_j) \leftarrow H(x_j + 1, y_j) + w_3 E(x_j, y_j)$$

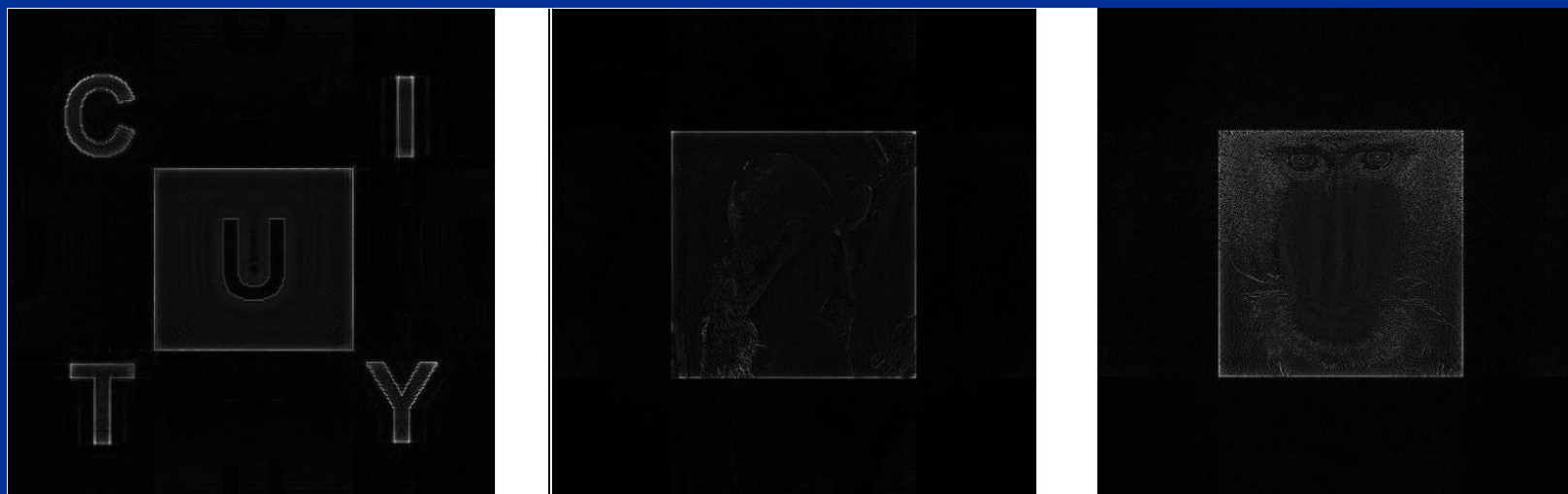
$$H(x_j + 1, y_j + 1) \leftarrow H(x_j + 1, y_j + 1) + w_4 E(x_j, y_j)$$



Uni-directional Error Diffusion (UERD) Phase Only Hologram



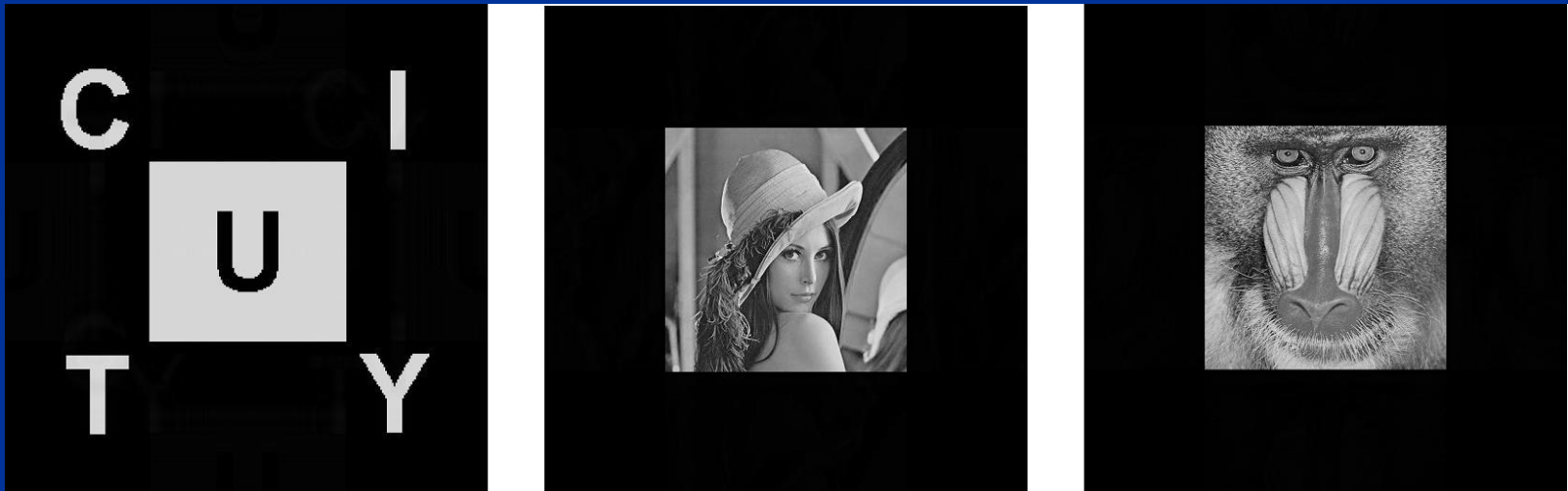
Original images



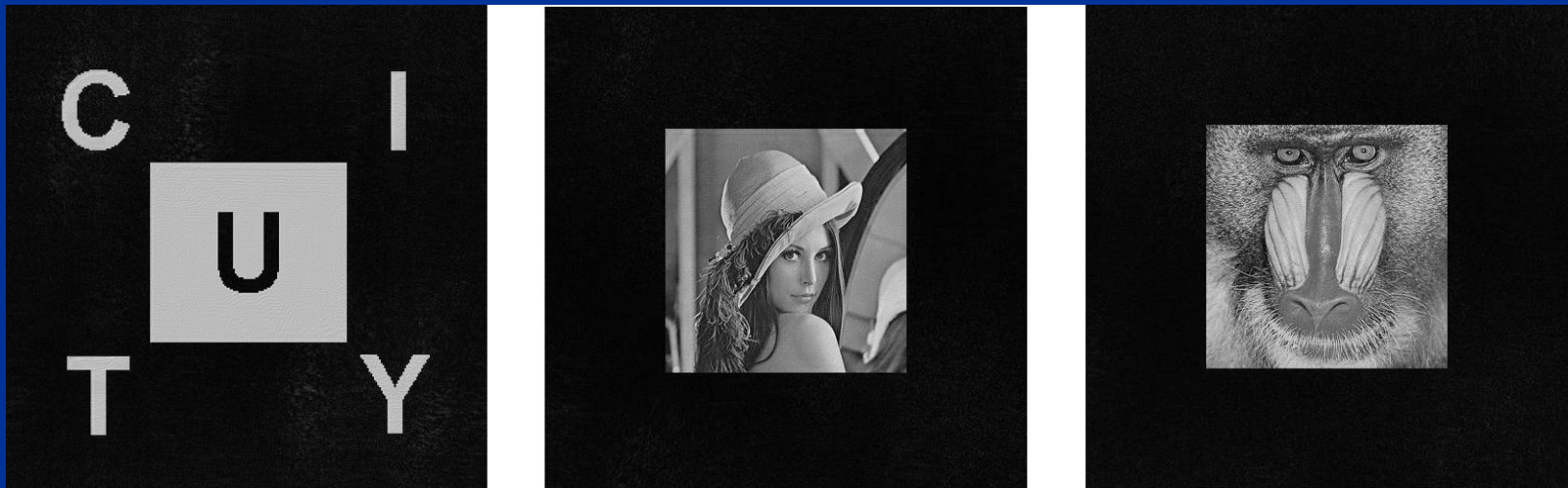
Reconstructed images from the phase components of the holograms



Uni-directional Error Diffusion (BERD) Phase Only Hologram



Original images

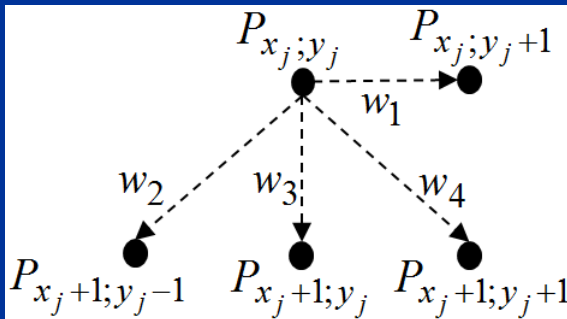


Reconstructed images from UERD holograms (noise is noted)

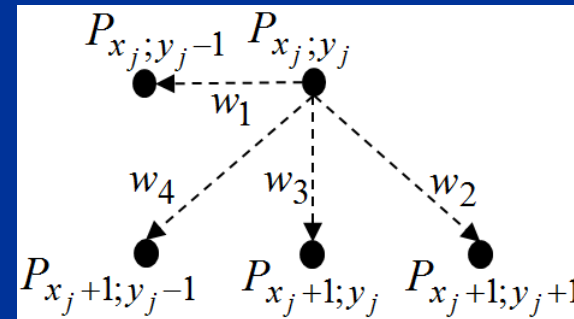


Bi-directional Error Diffusion (BERD) Phase Only Hologram

- Scan odd row of the complex hologram from left to right
- Scan even row of the complex hologram from right to left.
- Forced the magnitude of each scanned pixel to unity
- Diffuse error to the neighborhood, unvisited pixels



Odd rows

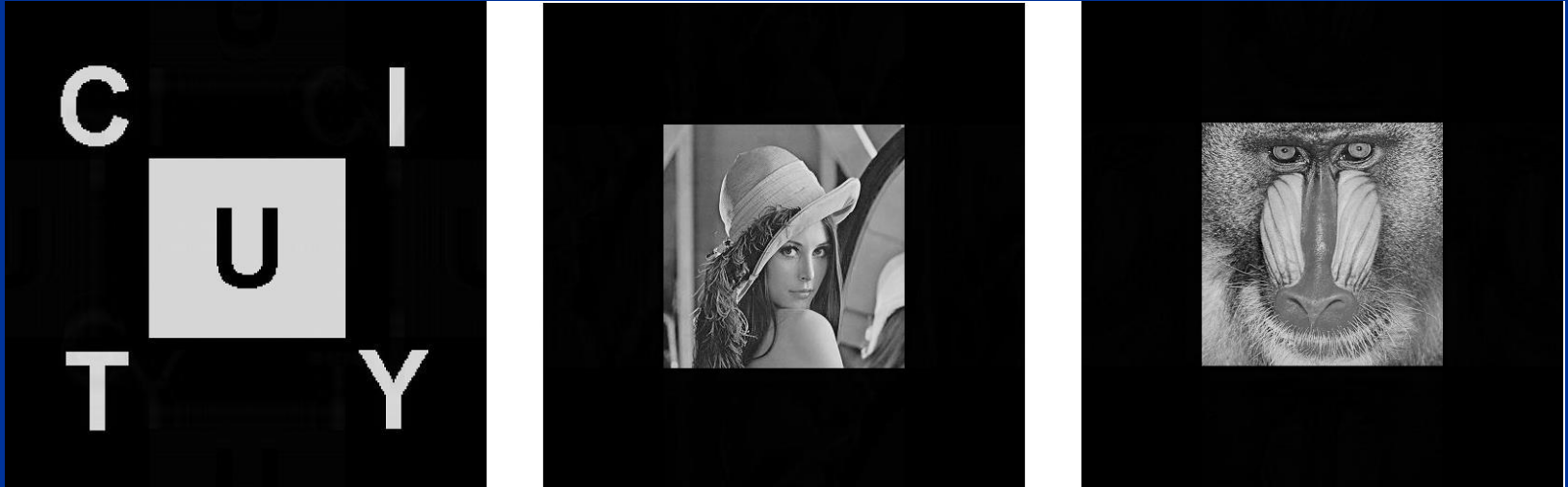


Even rows

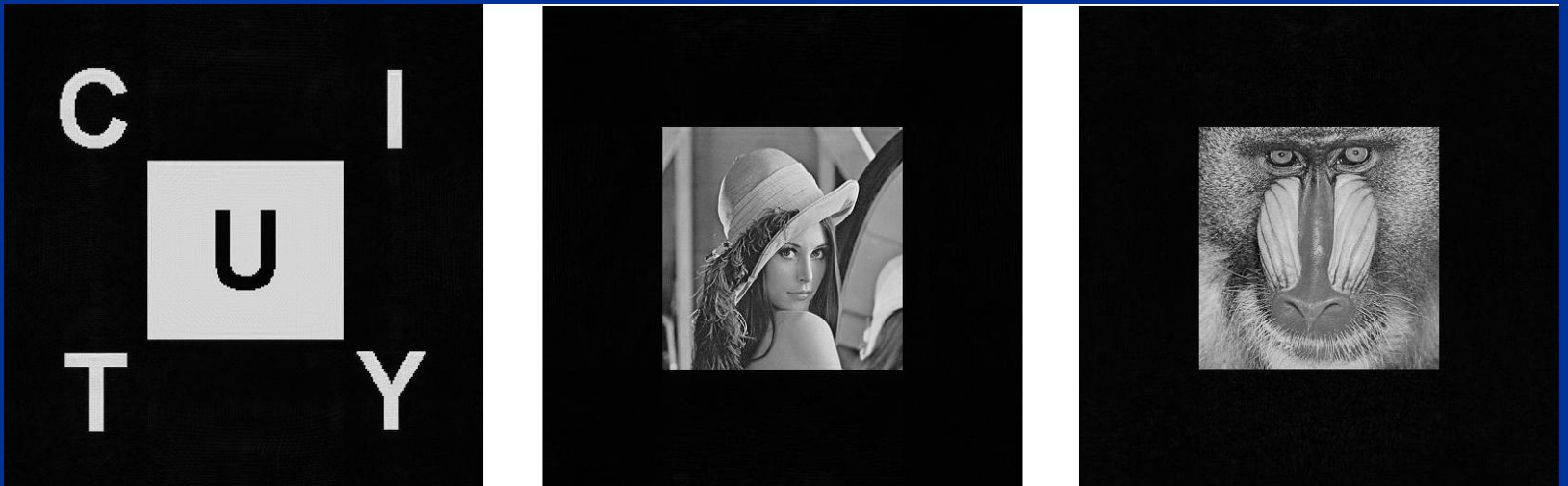
Partially de-correlates the error from the signal



Bi-directional Error Diffusion (BERD) Phase Only Hologram



Original images

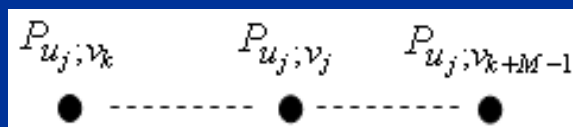
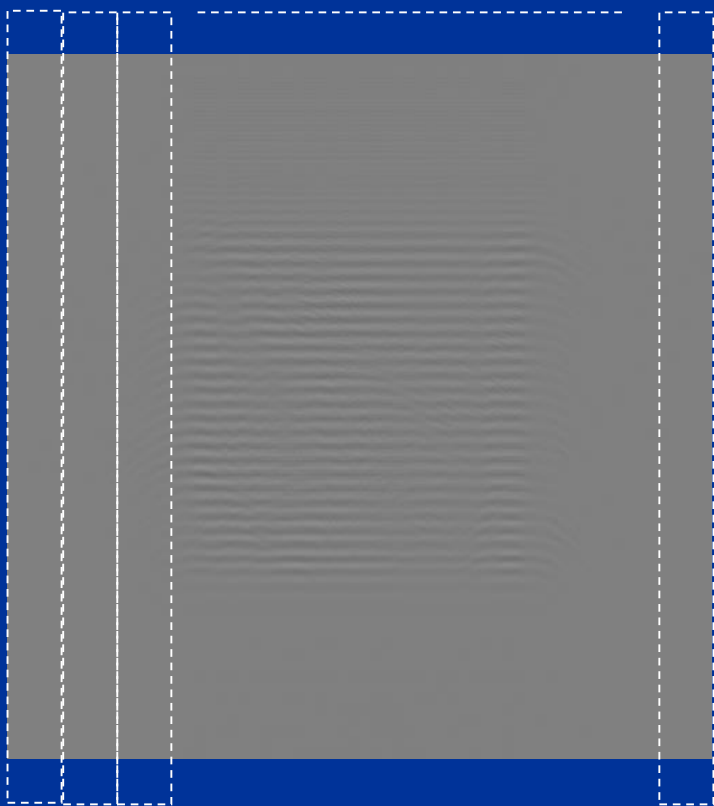


Reconstructed images from BERD holograms (noise is reduced)

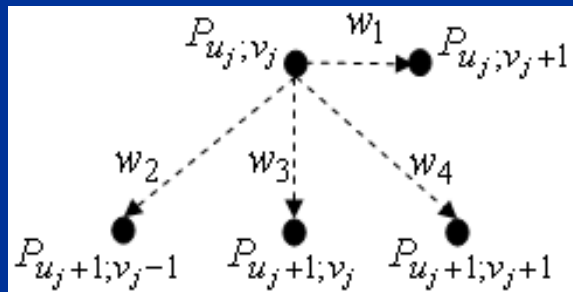


Localized error diffusion with redistribution (LERDR) phase only hologram

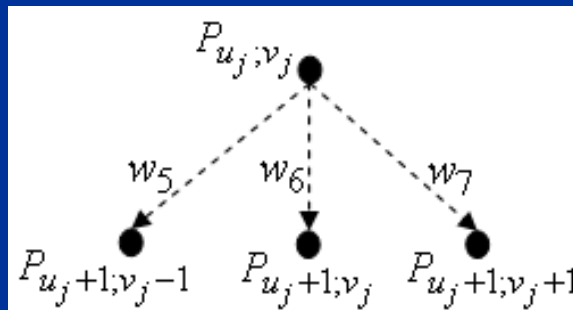
- Partition a hologram uniformly into vertical segments
- Apply localized error diffusion to each segment to convert the pixels into phase only value
- Apply low pass filtering to redistribute the error



A segment with M pixels



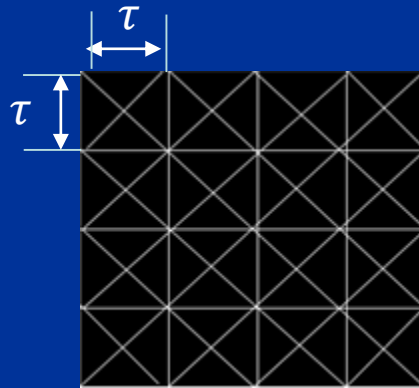
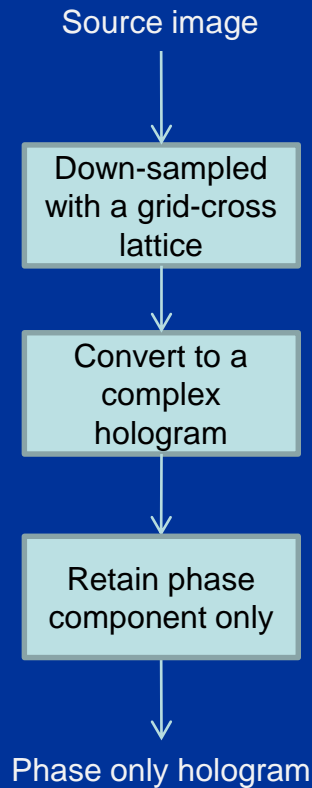
If not the last pixel, force the magnitude to unity, and distribute the error to the 4 neighboring pixels



For the last pixel, force the magnitude to unity, and distribute the error to the 3 neighboring pixels below it



Sampled Phase Only Hologram



$$S(x, y) = \bigcup_{k=0}^3 S_k(x, y)$$

$$S_0(x, y) = \begin{cases} 1 & \text{if } x \% \tau = 0 \\ 0 & \text{otherwise} \end{cases},$$

$$S_1(x, y) = \begin{cases} 1 & \text{if } y \% \tau = 0 \\ 0 & \text{otherwise} \end{cases},$$

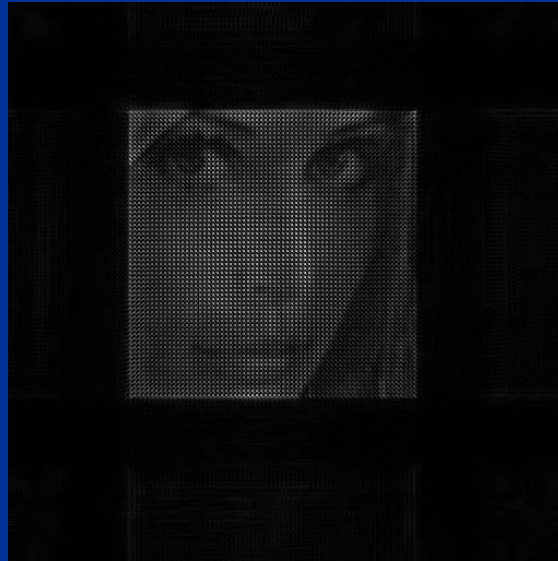
$$S_2(x, y) = \begin{cases} 1 & \text{if } (x \% \tau) = (y \% \tau) \\ 0 & \text{otherwise} \end{cases},$$

$$S_3(x, y) = \begin{cases} 1 & \text{if } (x \% \tau) = \tau - (y \% \tau) \\ 0 & \text{otherwise} \end{cases}.$$



Sampled Phase Only Hologram

Evaluation on reconstructed images



Fast, only involves a down-sampling process.

The reconstructed image is bright with favorable visual quality.

On the down-side, a texture is overlaid onto the reconstructed image.



Case study on a new method for holographic projection

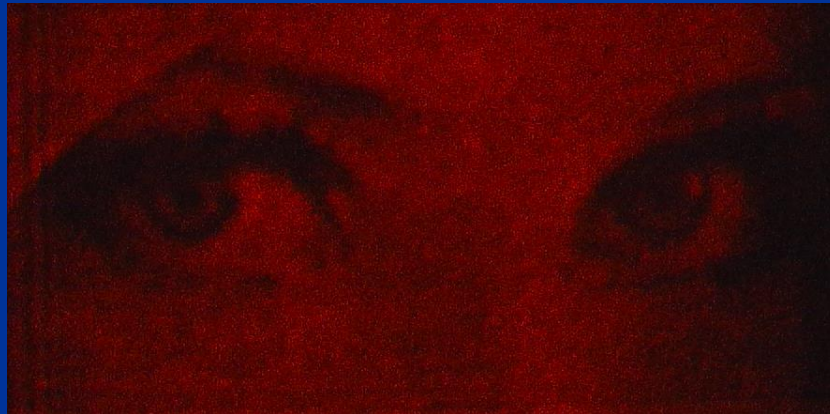
Optical reconstruction setup





Sampled Phase Only Hologram

Optical reconstructed images of a hologram representing single depth image.



The down-sampling texture is not prominent.



Case study on a new method for holographic projection

Optical reconstructed images of a hologram representing a double depth image.



Easy to assign different focal length to different part of the projected image

Projection can be adaptive to screen geometry

