Backgorund: Digital Communication Systems

In this note, we will review the basic transmitter and receiver principles in digital communication systems. The discussions will provide useful background for this course.

Binary phase shift keying (BPSK) based on a cos function

With BPSK modulation, the transmitted signal in $(kT, (k+1)T)$ is given by

$$s(t) = ax_k\cos(2\pi f_c t). \qquad (1)$$

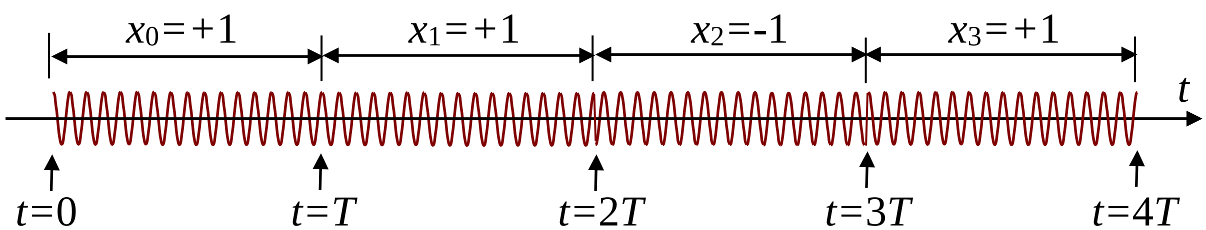
where a is used for power control and x_k represents a bit of information 0 or 1. For example:

$x_k = +1$ represents information 0, and

$x_k = -1$ represents information 1.

This is called a bipolar format of a binary bit.

The following is an example for four bits in time duration [0, 4T]. Pay attention to the phase jumps at time instances $t = 2T$ and $t = 3T$. This is caused by the change of sign of x_k at these points.

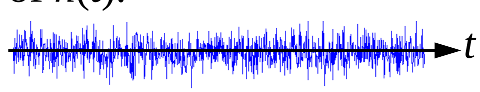


Assume that the channel has an attenuation factor ρ and a delay factor τ. The received signal is

$$r(t) = \rho a x_k\cos(2\pi f_c(t-\tau)) + n(t)$$

$$= \rho a x_k[\cos(2\pi f_c t)\cos(2\pi f_c \tau) + \sin(2\pi f_c t)\sin(2\pi f_c \tau)] + n(t). \quad (2)$$

where $n(t)$ is an additive white Gaussian noise (AWGN) with zero mean. The following is an illustration of $n(t)$.

AWGN noise $n(t)$ 

The following illustrates the received signal affected by AWGN noise.

Some useful equations:

The following equations are useful in understading the pricniples of BPSK and QPSK mentioned later.

In all cases, we assume that f is very large. In practice, f represents carrier frequency. A typical range of f is 1GHz ~ 5GHZ (i.e., 10^9 Hz to 5×10^9 Hz) for the 4G systems.

$$\left|\int_a^b \cos(2\pi ft)dt\right| = \frac{1}{2\pi f}|\sin(2\pi fb)-\sin(2\pi fa)| \le \frac{2}{2\pi f} \approx 0 \qquad (3a)$$

$$\left|\int_a^b \sin(2\pi ft)dt\right| = \frac{1}{2\pi f}|\left(-\cos(2\pi fb)\right)-\left(-\cos(2\pi fa)\right)| \le \frac{2}{2\pi f} \approx 0 \qquad (3b)$$

$$\int_a^b \sin(2\pi ft)\cos(2\pi ft)dt = 0.5\int_a^b \sin(2\cdot 2\pi ft)dt \approx 0 \qquad (3c)$$

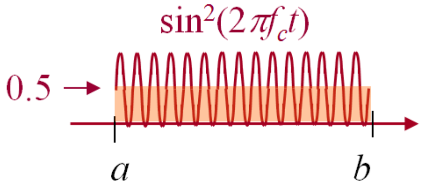
The following graphic illustration helps to understand the above relationships.

Furthermore, we have the following useful approximations:

$$\int_a^b \sin^2(2\pi ft)dt = 0.5\int_a^b (1-\cos(2\cdot 2\pi ft))dt \approx 0.5(b-a) \qquad (3d)$$

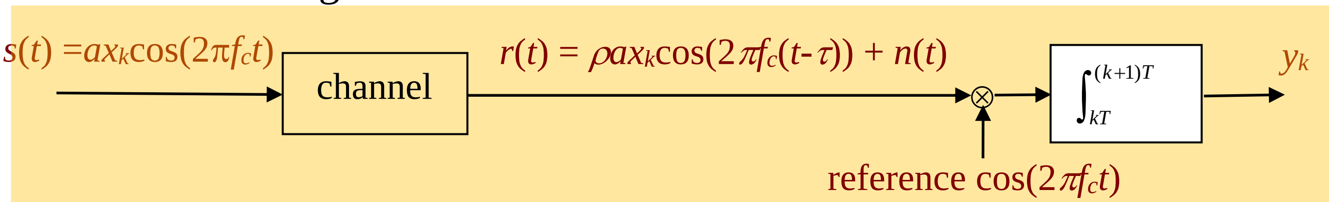
$$\int_a^b \cos^2(2\pi ft)dt = 0.5\int_a^b (1+\cos(2\cdot 2\pi ft))dt \approx 0.5(b-a) \qquad (3e)$$

Again, the above relationships can be understood using the following graph.



<u>Correlation receiver</u>

The following shows a correlation receiver to estimate x_k.



We repeat (2) as:

$$r(t) = \rho a x_k[\cos(2\pi f_c t)\cos(2\pi f_c \tau) + \sin(2\pi f_c t)\sin(2\pi f_c \tau)] + n(t). \quad (4)$$

Define $$\eta_k = \int_{kT}^{(k+1)T} n(t)\cos(2\pi f_c t)dt \quad . \quad (5)$$

From (3), we have

$$\int_{kT}^{(k+1)T} \cos^2(2\pi f_c t)dt \approx 0.5T \ ,$$

$$\int_{kT}^{(k+1)T} \sin(2\pi f_c t)\cos(2\pi f_c t)dt \approx 0 \ ,$$

The output y_k in the above receiver can then be expressed as

$$y_k = 0.5\rho a T x_k \cos(2\pi f_c \tau) + \eta_k \quad (6)$$

Assume that τ is known. We estimate x_k using y_k and the following rule:

$x_k = +1$ if y_k has the same sign as $0.5\rho a T\cos(2\pi f_c \tau)$ (7a)

$x_k = -1$ otherwise. (7b)

Note that η_k is unknown, which may cause detection error. We model η_k as Gaussian distributed with zero mean and variance = $0.5^2 T N_0$, where N_0 is called single-sided channel noise density. N_0 is a measurement of channel noise level.

The performance of the above receiver is determined by the following signal to noise ration (SNR)

$$SNR = \frac{\left(0.5\rho a T\cos(2\pi f_c \tau)\right)^2}{\text{variance of } \eta_k} = \frac{0.5^2 \rho^2 a^2 \cos^2(2\pi f_c \tau) T^2}{0.5^2 N_0 T}$$

$$= 2\rho^2 \cos^2(2\pi f_c \tau)(E_b / N_0) . \quad (8)$$

where E_b=$0.5a^2T$ is the transmitted energy per bit (based on (3b)). Note that SNR is affected by E_b, path gain ρ^2 and delay τ.

In-phase and quadrature signal components

The signal above is modulated by $\cos(2\pi f_c t)$. This is usually referred to as the in-phase component. If $\sin(2\pi f_c t)$ is used for modulation, as discussed below, the related signal is called the quadrature compoenent.

If the channel delay τ is not zero, then part of the in-phase componenet is turned into a quadrature componenet as (see (2))

$$\rho a x_k \sin(2\pi f_c t)\sin(2\pi f_c \tau).$$

Such component is orthogonal to the reference $\cos(2\pi f_c t)$ (see the figure and (3)) since

$$\int_a^b \sin(2\pi ft)\cos(2\pi ft)dt \approx 0.$$

The related energy is not utilized, which reduces energy efficiency.

Binary phase shift keying (BPSK) based on a sin function

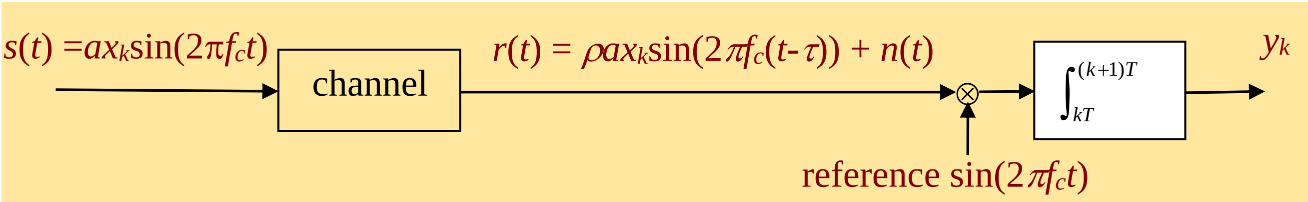
We can modulate a signal using a sin function as

$$s(t) = a x_k \sin(2\pi f_c t). \quad (9)$$

The received signal is

$$r(t) = \rho a x_k \sin(2\pi f_c(t-\tau)) + n(t)$$
$$= \rho a x_k[\sin(2\pi f_c t)\cos(2\pi f_c \tau) - \cos(2\pi f_c t)\sin(2\pi f_c \tau)] + n(t). \quad (10)$$

The following shows a correlation receiver to estimate x_k. Note that the reference is $\sin(2\pi f_c t)$ now.



Define $\eta_k = \int_{kT}^{(k+1)T} n(t)\sin(2\pi f_c t)dt$. (11)

We can write the output y_k in the above receiver as

$$y_k = 0.5\rho a T x_k \cos\left(2\pi f_c \tau\right) + \eta_k. \quad (12)$$

We can again estimate x_k based on y_k. The rule is similar to (7). Again, part of the received signal, i.e., $\rho a x_k \cos(2\pi f_c t)\sin(2\pi f_c \tau)$, is not utilized.

Quadrature phase shift keying (QPSK)

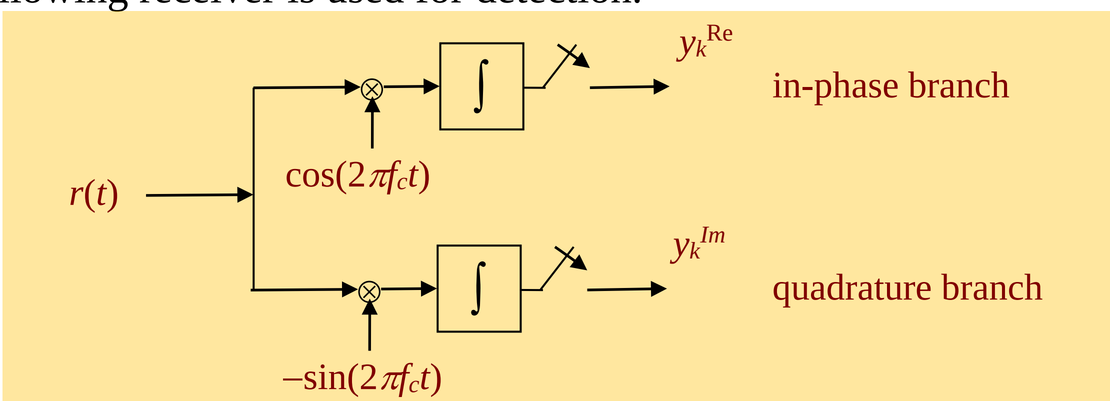
We can increase transmission speed as well as energy efficiency using both in-phase and quadrature componenets. This is referred to as quadrature phase shift keying (QPSK). The transmitted signal is

$$s(t) = ax_k^{\mathrm{Re}} \cos(2\pi f_c t) - ax_k^{\mathrm{Im}} \sin(2\pi f_c t)\,. \tag{13}$$

The received signal is

$$\begin{aligned} r(t) &= \rho a x_k^{\mathrm{Re}} \cos\left(2\pi f_c \left(t-\tau\right)\right) - \rho a x_k^{\mathrm{Im}} \sin\left(2\pi f_c \left(t-\tau\right)\right) + n(t) \\ &= \rho a[x_k^{\mathrm{Re}} \cos(2\pi f_c t)\cos(2\pi f_c \tau) + x_k^{\mathrm{Re}} \sin(2\pi f_c t)\sin(2\pi f_c \tau)] \\ &\quad + \rho a[-x_k^{\mathrm{Im}} \sin(2\pi f_c t)\cos(2\pi f_c \tau) + x_k^{\mathrm{Im}} \cos(2\pi f_c t)\sin(2\pi f_c \tau)] + n(t) \end{aligned} \tag{14}$$

The following receiver is used for detection.



Similar to the discussions earlier, we can shown the following.

$$y_k^{\mathrm{Re}} = 0.5\rho a T \left(x_k^{\mathrm{Re}} \cos\left(2\pi f_c \tau\right) + x_k^{\mathrm{Im}} \sin\left(2\pi f_c \tau\right)\right) + \eta_k^{\mathrm{Re}}, \tag{15a}$$

$$y_k^{\mathrm{Im}} = 0.5\rho a T \left(-x_k^{\mathrm{Re}} \sin\left(2\pi f_c \tau\right) + x_k^{\mathrm{Im}} \cos\left(2\pi f_c \tau\right)\right) + \eta_k^{\mathrm{Im}}. \tag{15b}$$

Now we introduce notations:

$$x_k = x_k^{\mathrm{Re}} + jx_k^{\mathrm{Im}}, \tag{16a}$$

$$y_k = y_k^{\mathrm{Re}} + jy_k^{\mathrm{Im}}, \tag{16b}$$

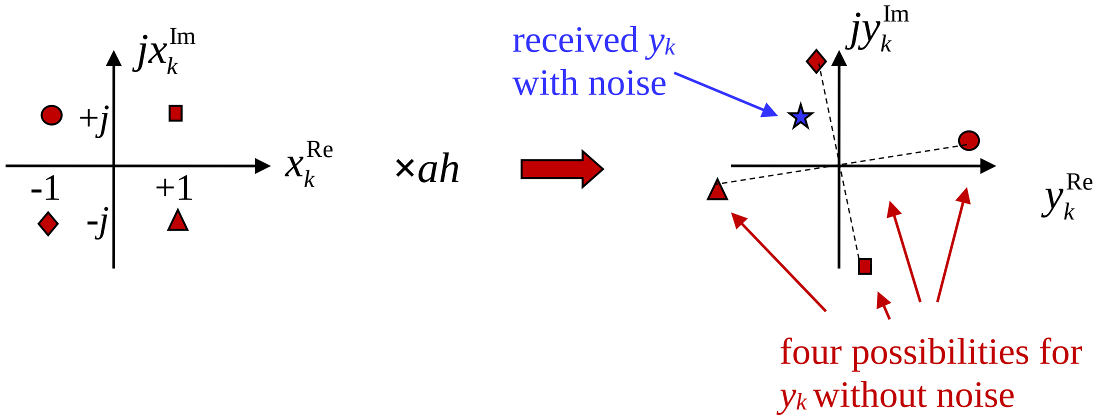
$$h = 0.5T\rho\left(\cos\left(2\pi f_c \tau\right) - j\sin\left(2\pi f_c \tau\right)\right), \tag{16c}$$

$$\eta_k = \eta_k^{\mathrm{Re}} + j\eta_k^{\mathrm{Im}}. \tag{16d}$$

Then (15) and (16) lead to a very simple expression:

$$y_k = ahx_k + \eta_k \tag{17}$$

Detection for QPSK



We can carry out detection using the above graphis illustration. Assume that h_k is know. Without noise η_k, we can find four noiseless possibilities for y_k. With noise η_k, the actual y_k is not on these four points. We can use the minimum Euclidian distance to find the best estimate. For example, for the received signal shown by the "star" above, the best estimate is the "diamond" since it is closest to the "star".

Notes:

- The transmitted and received signals are all real. The complex notations are used only for simplicity. We call $x_k = x_k^{\mathrm{Re}} + jx_k^{\mathrm{Im}}$ a QPSK modulated symbol. It is a simplified notation for the actual signal $s(t) = ax_k^{\mathrm{Re}} \cos(2\pi f_c t) - ax_k^{\mathrm{Im}} \sin(2\pi f_c t)$.
- The received signal consists of two parts, i.e., y_k^{Re} and y_k^{Im} . They are both real signals. Complex notation is again used only for simplicity.
- Since exp(jz) = cos(z) + $j$sin(x). We can write h in a phasor form:

$$h = 0.5T\rho\exp\left(-j2\pi f_c\tau\right).$$

  Here ρ is refered to as channel magnitude, $|\rho|^2$ as channel power gain (or simply channel gain) and τ as channel delay. Both ρ and τ are determined by channel only. A constant factor 0.5T is introduced by the intergration operation at the receiver.
- In the QPSK receiver, both in-phase and quadrature compoenets can be fully utilized. Such a structure can also be used to detect a BPSK signal with improved power efficiency.

The Q-Function

In probability theory, the Gaussian distribution is considered the most important probability distribution in statistics. This is why it is also called normal distribution. It means that most things follow this distribution “normally”.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}$$

For example, temperature, water level, incomes, exam results,…

Define

$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-t^2}{2}} dt$$

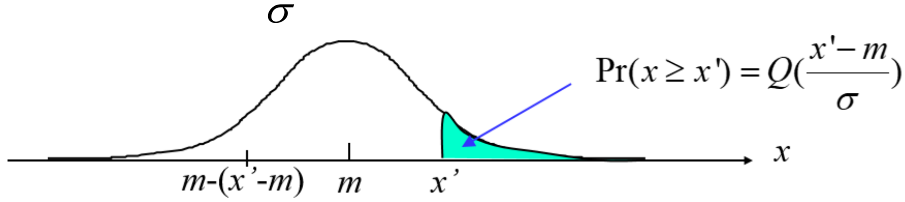
Then for a Gaussian distribution,

$$\Pr(x \geq x') = \int_{x'}^{\infty} p(x)dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{x'}^{\infty} e^{\frac{-(x-m)^2}{2\sigma^2}} dx \overset{t=\frac{x-m}{\sigma}}{=} \frac{1}{\sqrt{2\pi}} \int_{\frac{x'-m}{\sigma}}^{\infty} e^{\frac{-t^2}{2}} dt = Q(\frac{x'-m}{\sigma})$$

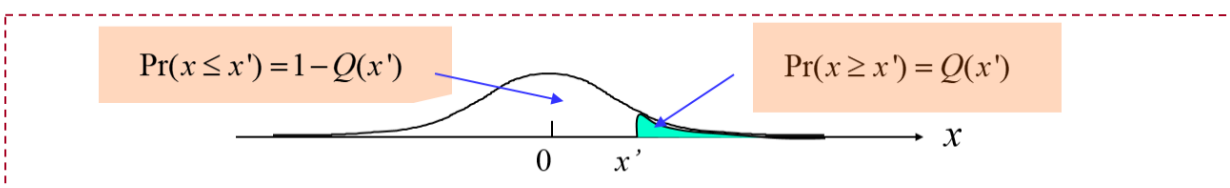
or simply

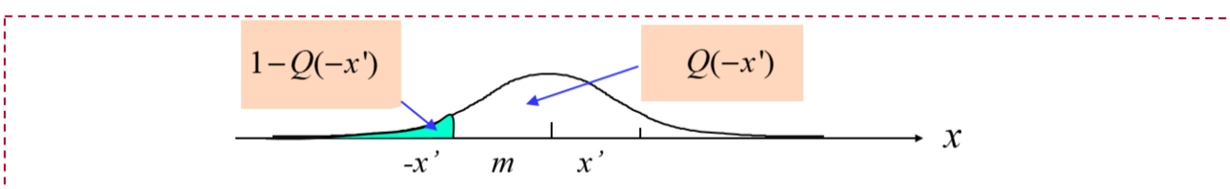
$$\Pr(x > x') = Q(\frac{x'-m}{\sigma})$$

Q1) Pr(x<x') = ?
Q2) Pr(x<m-(x'-m)) = ?
Q3) Pr(x>m-(x'-m)) = ?



Some useful relationships are as follows:





Therefore 1-Q(-x) = Q(x)
i.e., Q(-x) + Q(x) =1

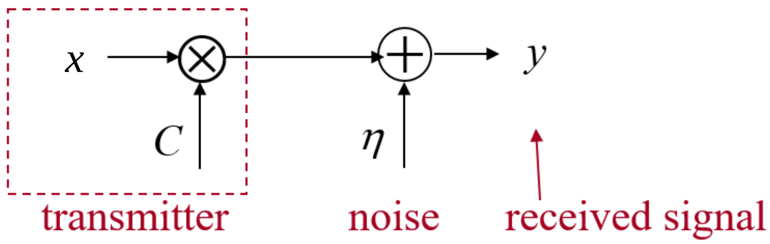
Error events in a communication system

We now consider the impact of noise on detection error in a communication system. We will assume that the noise is Gaussian. Consider the following system model.

$$y = xC + \eta.$$

Here

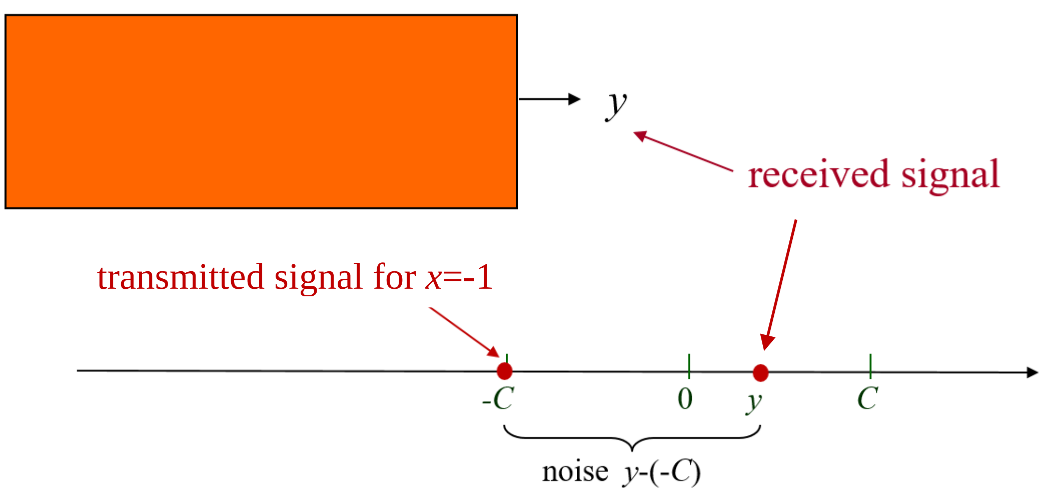
- $x=\pm 1$ represents a bit of information,
- C is used to adjust transmission power (noting that power = C^2),
- h is a Gaussian distributed variable with mean = 0 and variance = σ^2.



The receiver does not know x (otherwise why do we bother to transmit). How can receiver find x? Let us guess x as follows:

- If $y>0$, we guess that $x = 1$.
- If $y\le 0$, we guess that $x = -1$.

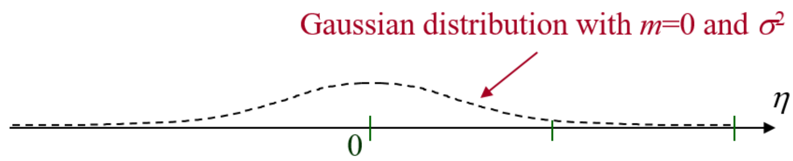
Is this method 100% correct? Of course not. Then what is the error probability?



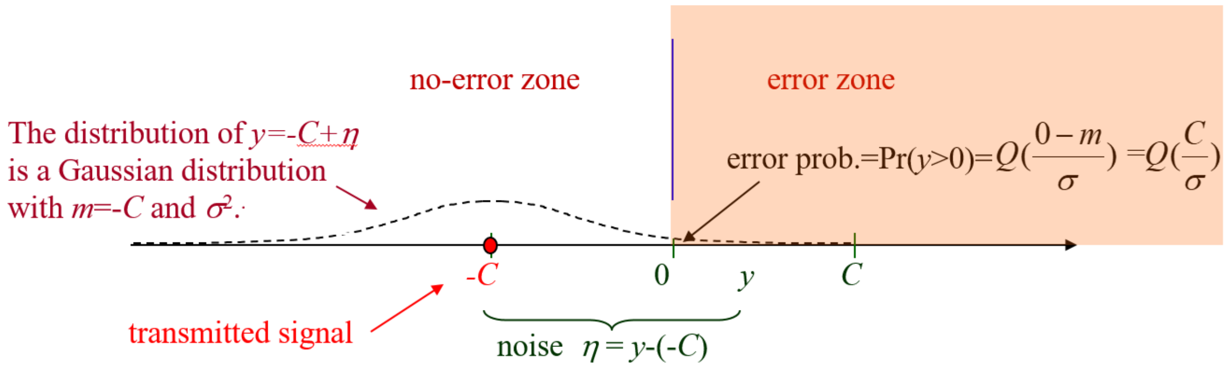
Bit error rate (BER)

Received signal : $y = xC + \eta.$

Noise distribution:



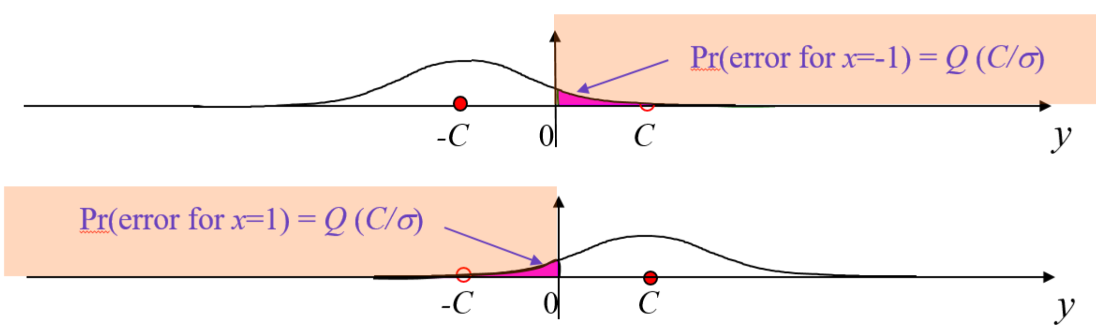
The distribution of y when x=-1:



Pr(error for x=1) = $Q(C/\sigma)$

Pr(error for x=-1) = $Q(C/\sigma)$

BER = (overall) bit error rate = $Q(C/\sigma)$



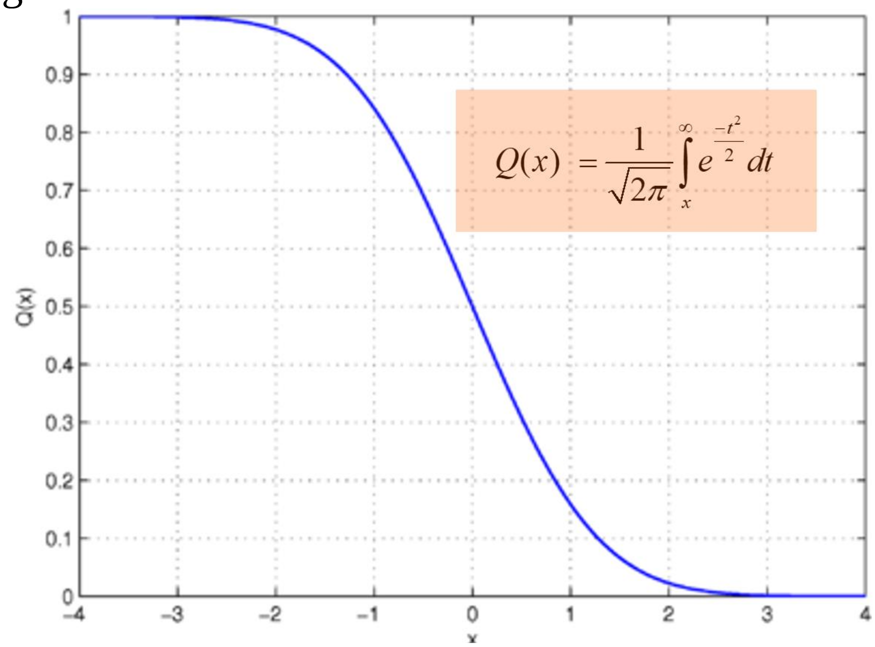
Is the average bit error rate (i.e. BER) $2Q(C/s)$. No. The probability of x= 1 is 0.5 and the probability of x =-1 is also 0.5. Hence the BER is still $Q(C/s)$.

Bit error rate and SNR

From the above discussions,

$$\text{BER} = Q\left(\frac{C}{\sigma}\right) = Q\left(\sqrt{\frac{C^2}{\sigma^2}}\right).$$

Here C^2/σ^2 is referred to as SNR. Clearly, we can see that BER is a decreasing function of SNR.



Summary

With QPSK modulation, the signals involved can be expressed using the following concise notations:

$$y_k = y_k^{\text{Re}} + j y_k^{\text{Im}} = h \cdot (a x_k) + \eta_k,$$

$$h = h_k^{\text{Re}} + j h_k^{\text{Im}} = 0.5 T \rho \exp(-j 2\pi f_c \tau),$$

$$x_k = x_k^{\text{Re}} + j x_k^{\text{Im}},$$

$$\eta_k = \eta_k^{\text{Re}} + j \eta_k^{\text{Im}}.$$

Information is carried in x_k. The total transmitted power is $|a|^2$ for both real and imaginary parts. (Recall that the average power of cos or sin are both 0.5.) Channel power gain is ρ^2 and channel delay is τ.

For a BPSK system, BER is a decreasing function of SNR, given by $\text{BER} = Q\left(\sqrt{C^2/\sigma^2}\right)$.