# Chapter 3

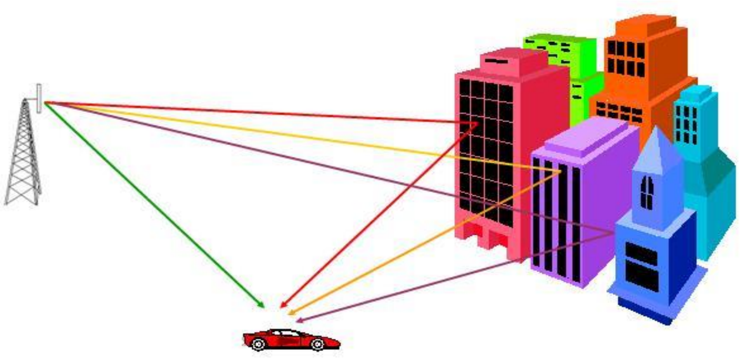
# Statistical Multipath Channel Models

# Introduction

# Introduction

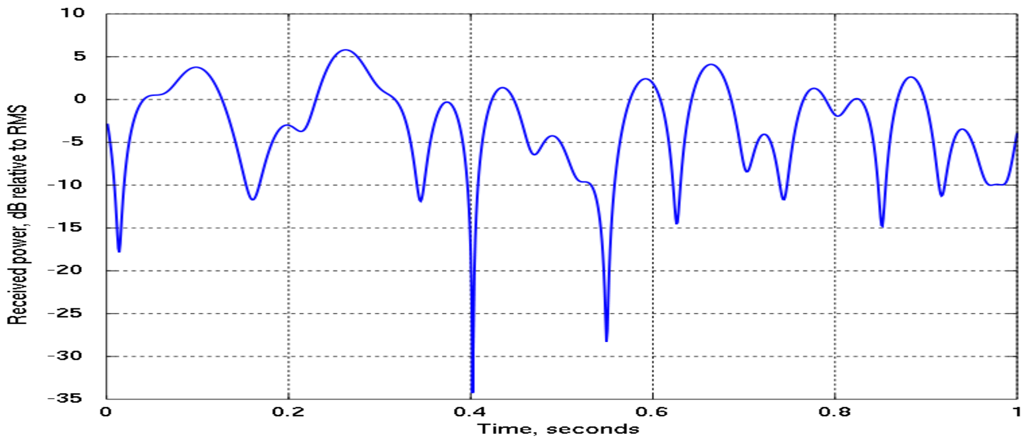
Path loss and shadowing discussed in Chapter 2 are caused by large-scale characteristics of the environment. Therefore they are usually referred to as “large-scale” fading. (Some people prefer to use “large scale fading” only for shadowing.)

Radio wave reflection occurs at the earth surface as well as other objects. The location and the surface conditions of the objects are usually unknown to the transmitter and the receiver. The characteristic of the signals in mobile channel is a very complicated problem. This results in “small-scale” fading.



# Introduction

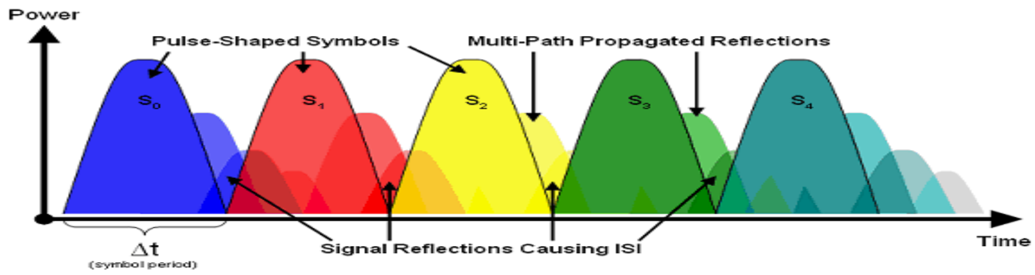
There are several effects of small-scale fading. The first is Rayleigh fading caused by multiple reflections of the transmitted signals arriving at the receiver at slightly different times. The reflections gave different delays that may cause both amplitude and phase distortions. They may enhance each other or cancel each other. Thus signal strength fluctuates at different locations.




The second is that the carrier frequency of the received signal becomes randomly varying within a range determined by the so-called Doppler frequency.

# Introduction

The third occurs when the dimensions of the reflection objects are relatively large (though still smaller than that for large scale fading). In this case, due to the time difference among multiple reflections, the transmitted pulse can be “expanded” when it arrives at the receiver. When delay difference further increases, a receiver may receive multiple replicas of each transmitted pulse. In a digital system, such effect may cause inter-symbol interference (ISI).



# Introduction

In what follows, we will discuss the impact of relatively small-scale characteristics of the environment. The effect is sometimes referred to as small-scale fading.

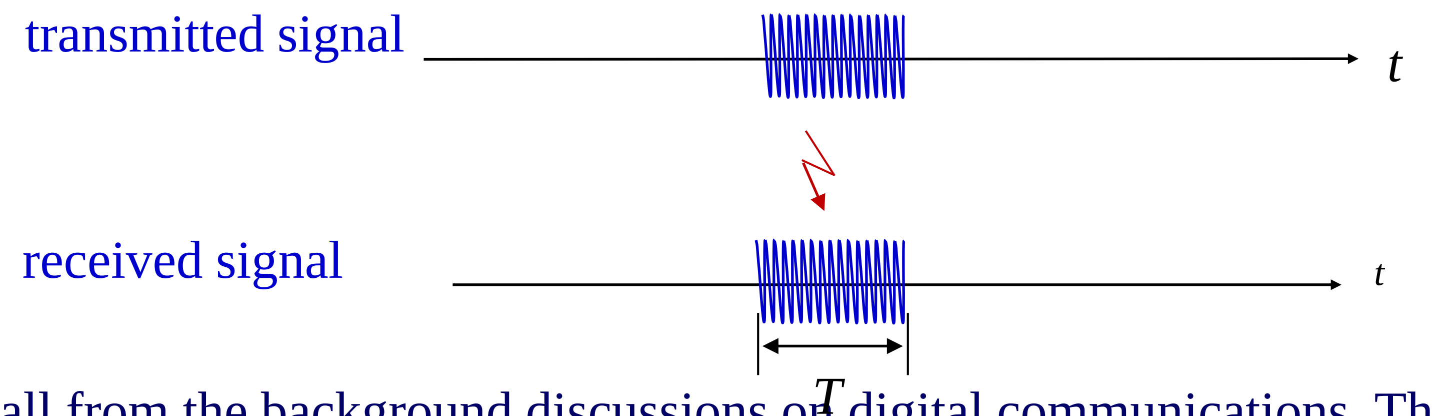
This Chapter includes the following parts.

3.1 Digital channel characterization
3.2 Rayleigh and Rician fading
3.3 Overall Fading Effect
3.4 Doppler effect
3.5 Time domain statistical channel modeling
3.6 Delay spread
3.7 Frequency domain statistical channel modeling
3.8 Channel simulation

# 3.1 Digital channel characterization

# Single-path channel

The following is an illustration of a simple single-path channel model.




Recall from the background discussions on digital communications. The transmitted signal is given by

$$s(t) = x_n^{\mathrm{Re}} \cos(2\pi f_c t) - x_n^{\mathrm{Im}} \sin(2\pi f_c t) \qquad (1)$$

# Single-path channel

The following notations are used:

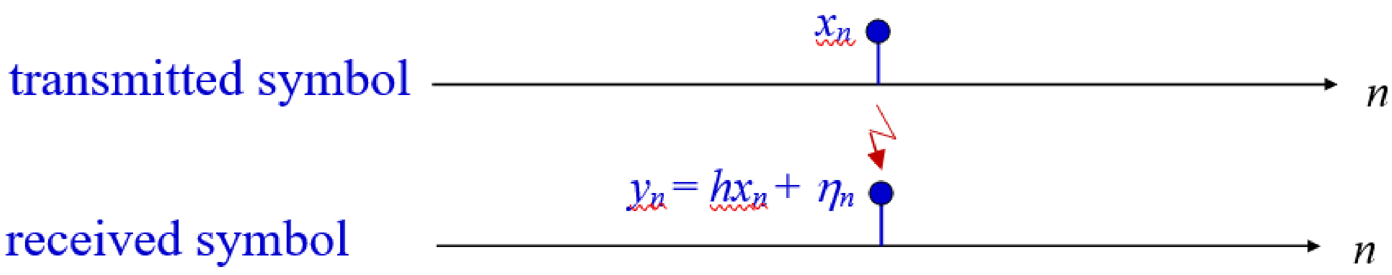
$$y_n = y_n^{\mathrm{Re}} + jy_n^{\mathrm{Im}} \qquad x_n = x_n^{\mathrm{Re}} + jx_n^{\mathrm{Im}} \tag{2.a}$$

$$h = h^{\mathrm{Re}} + jh^{\mathrm{Im}} \qquad \eta_n = \eta_n^{\mathrm{Re}} + j\eta_n^{\mathrm{Im}} \tag{2.b}$$

A single-path channel is characterized by a very simple expression:

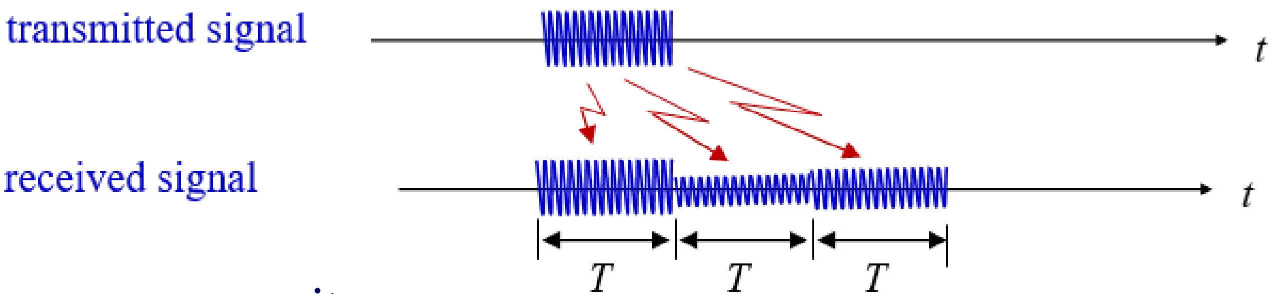
$$y_n = hx_n + \eta_n \tag{3}$$

Note that the above model does not cover channel only. It also includes the operation of a correlator receiver. Such a model is illustrated graphically below.



# Multi-path channel

With reflections, multiple replicas may arrive at the receiver for each transmitted symbol. We first assume that each path delay is given by $kT$, where $k$ is a non-negative integer.

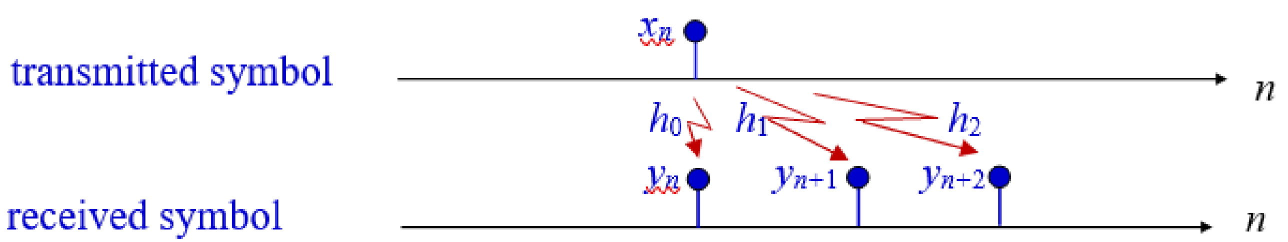



In this case, we can write

$$y_{n+k} = h_k x_n + \eta_{n+k} \qquad (4a)$$

or equivalently

$$y_n = h_k x_{n-k} + \eta_n \qquad (4b)$$



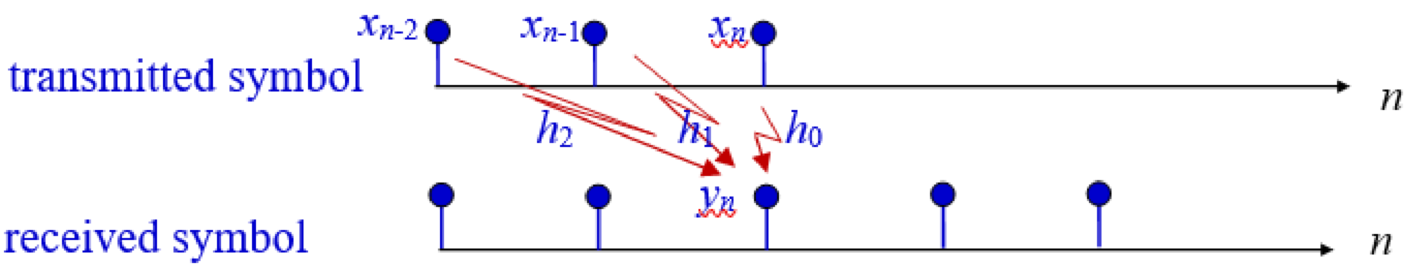
# Multi-path channel

The above is for one transmitted symbol. We now consider multiple transmitted symbols

$$\ldots x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots$$

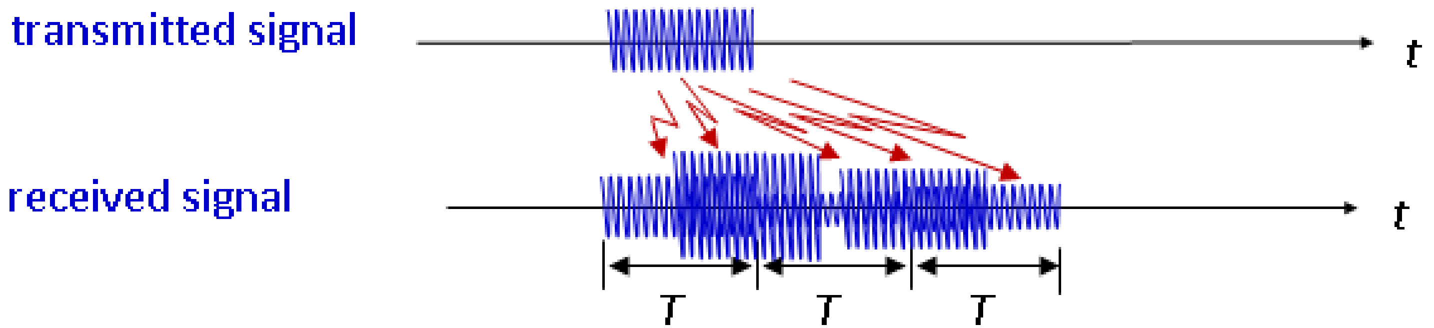
A received symbol is the sum of the responses of all the transmitted symbols. We can rewrite (4b) as

$$\begin{aligned} y_n &= h_0 x_n + h_1 x_{n-1} + h_2 x_{n-2} + \ldots + \eta_n \\ &= \sum_k h_k x_{n-k} + \eta_n . \end{aligned} \tag{5}$$

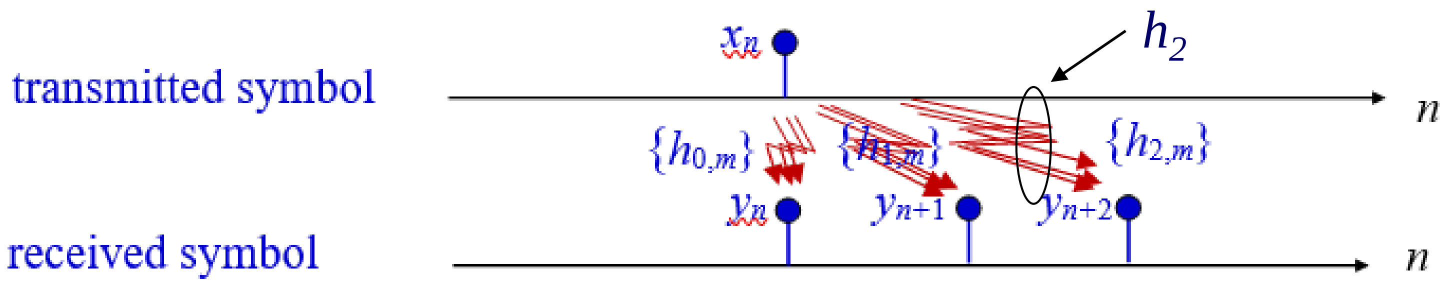


# General multi-path channel

We now consider a more general case. Assume that the delay for each path can be written as $mT$, where $m$ is not necessarily an integer. This is illustrated below.




The following shows the situation after a correlation receiver.



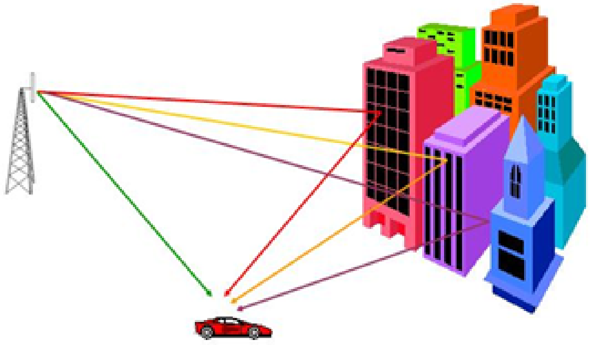
# General multi-path channel

Each $h_k$ includes the effect of several multipath reflections. It can be expressed as the sum of many terms, each representing a ray falling into period $(kT, (k+1)T)$:

$$h_k = h_k^{\mathrm{Re}} + jh_k^{\mathrm{Im}} = \sum_m h_{k,m}. \quad (6)$$

With multiple transmitted symbols, the channel model is the same as before:

$$y_n = \sum_k h_k x_{n-k} + \eta_n. \quad (7)$$



# Notes:

$$y_n = \sum_k h_k x_{n-k} + \eta_n .$$

(i) The above is for a time-invariant channel only. It can be written in in a more compact convolution form as follows:

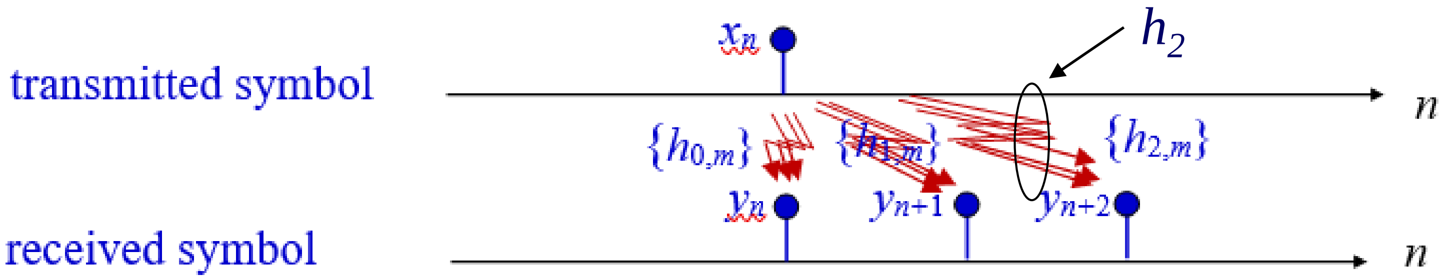
$$\boldsymbol{y} = \boldsymbol{h} \otimes \boldsymbol{x} + \boldsymbol{\eta}.$$

(ii) Each $h_k$ may include the effect of several reflections.

(ii) All variables can be complex. They represent the in-phase and quadrature signals carried by cosine and sine waveforms. The actual signals can be found by taking the real and imaginary parts.

# 3.2 Rayleigh and Rician fading

# Rayleigh fading




Repeat (6):

$$h_k = h_k^{\mathrm{Re}} + jh_k^{\mathrm{Im}} = \sum_m h_{k,m}. \qquad (8)$$

From central limit theorem, the real and imaginary parts of $h_k$ are approximately Gaussian distributed. Let their mean = 0 and variances $=\sigma^2$. Define a new random variable

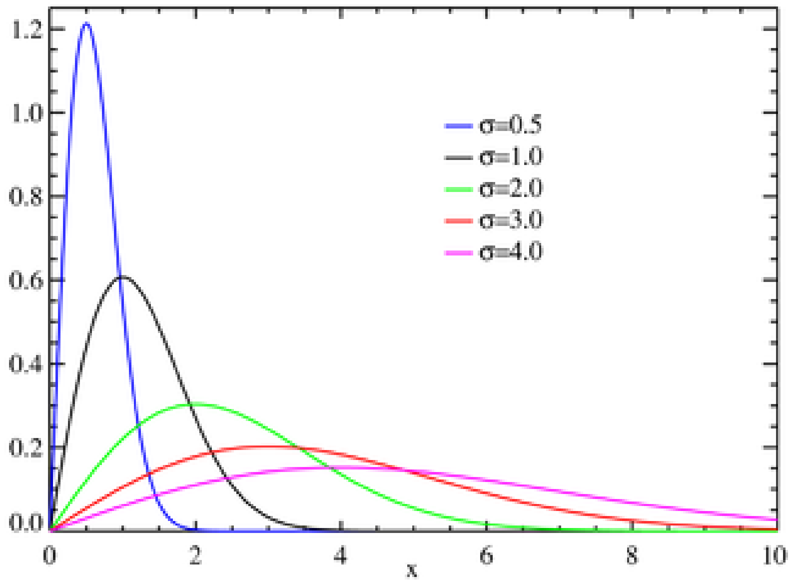
$$r_k = \sqrt{\left(h_k^{\mathrm{Re}}\right)^2 + \left(h_k^{\mathrm{Im}}\right)^2}. \qquad (9)$$

We treat $r_k$ as a sample of a random variable $r$. When the real and imaginary parts of $h_k$ are both Gaussian, $r$ is Rayleigh distributed with PDF given by:

$$p_r(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \qquad (10)$$

# Rayleigh fading

The plot of $p_r(r)$ is shown below.




The mean value, mean power and variance of *r* are given by

$$E(r) = \sigma\sqrt{\pi / 2}$$

$$E(r^2) = 2\sigma^2$$

$$Var(r) = E(r^2) - (E(r))^2 = (2 - \frac{\pi}{2})\sigma^2$$

# Exponential distribution

Let $a$ and $b$ be two independent Gaussian random variables with mean=0 and variance = $\sigma^2$. The random variable

$$z = (a^2+b^2) = r^2$$

follows the exponential distribution. The PDF of $z$ is given below.

$$p_z(z) = \begin{cases} \frac{1}{2\sigma^2} e^{-\frac{z}{2\sigma^2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \qquad (11a)$$

A commonly used alternative expression is (with $\lambda=1/(2\sigma^2)$)

$$p_z(z) = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (11b)$$

# Notes

In the above, $h = h^{\mathrm{Re}} + \mathrm{j}h^{Im}$ is a random variable. Its real and imaginary parts $h^{\mathrm{Re}}$ and $h^{Im}$ are Gaussian distributed with the same mean and variance.

Here $h$ actually involves three distributions.

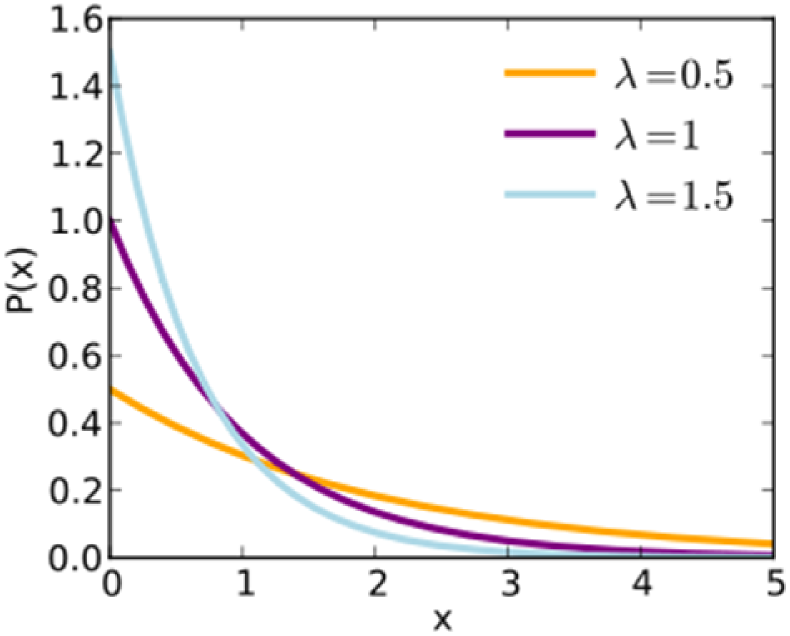
- The real and imaginary parts of $h$ are Gaussian distributed.
- The amplitude of $h$ is Rayleigh distributed.
- The power of $h$ is exponential distributed.

Traditionally, we say that $h$ represents Rayleigh fading.

Keep in mind that there are three distributions involved.

# Exponential distribution

Its plot is given below.



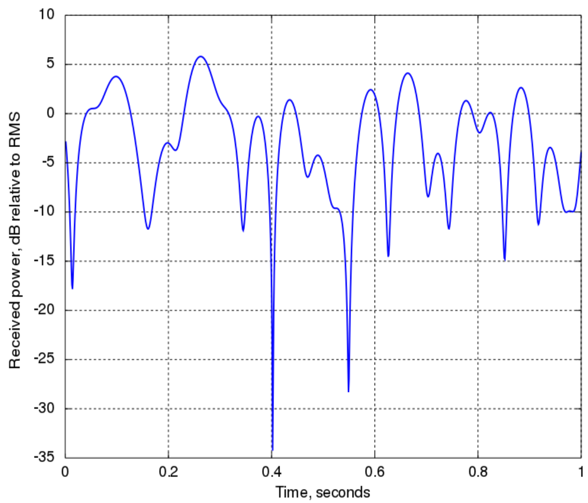

The mean value and mean power of $z$ is given by

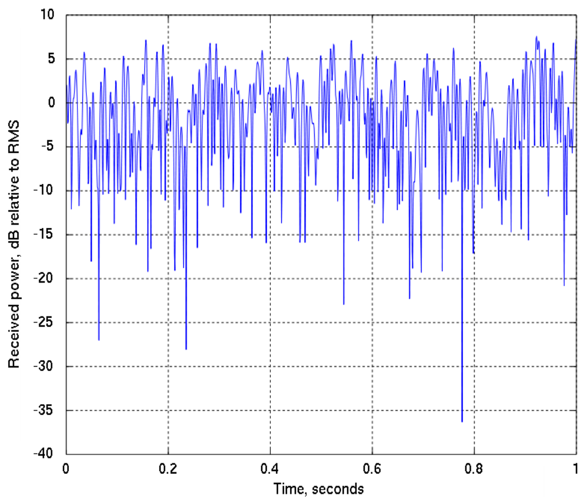
$$E(z) = E(r^2) = 2\sigma^2$$

$$Var(z) = E(z^2) - (E(z))^2 = 4\sigma^4$$

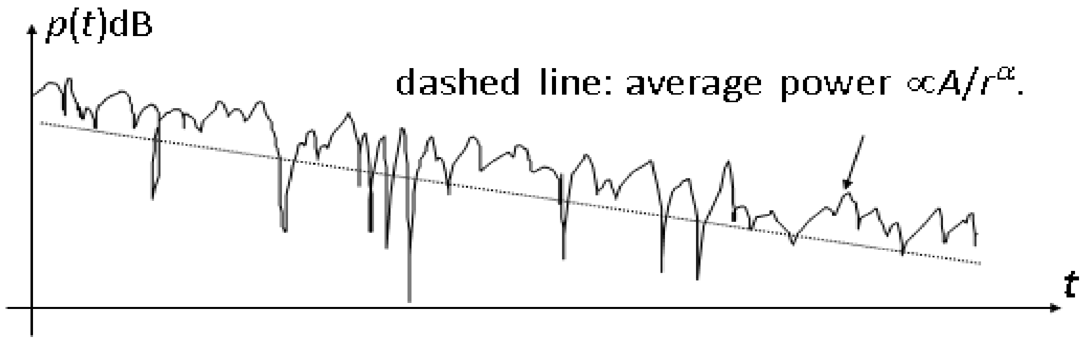
# Illustration on Rayleigh fading

Two signals with the Rayleigh fading effect on moving terminals are illustrated in the figures below. They are one second snapshots of received power levels with a maximum Doppler shift of 10 Hz (left) and 100Hz (right), respectively.

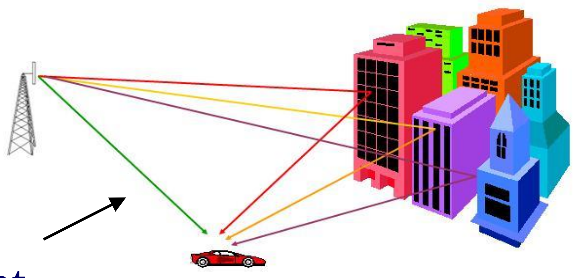


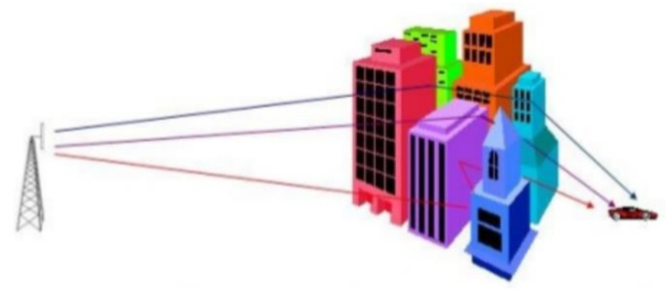




The figure shown below further takes path loss into account



# Rician fading





direct path
line of sight

If there is a direct path, there is a dominant one among different reflections. In this case the law of large numbers cannot be applied.

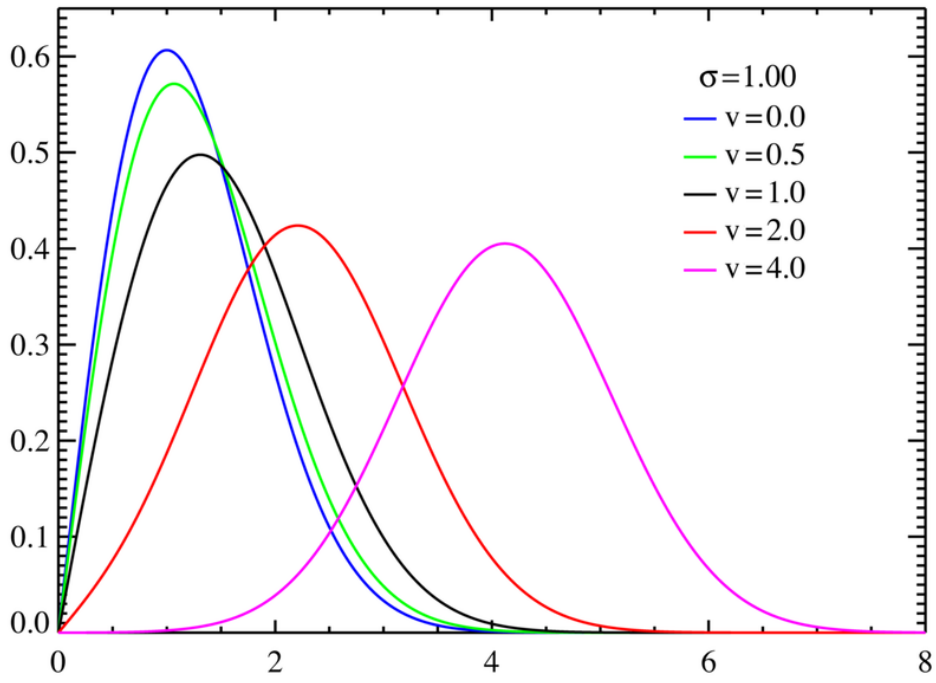
It can be shown that the amplitude of $h_k = \Sigma h_{k,m}$ follows a Rician distribution when there is a dominant term. The PDF of a Rician distributed variable is

$$p_r(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-\frac{(r^2+v^2)}{2\sigma^2}} \mathrm{I}_0\left(\dfrac{v\mathrm{r}}{\sigma^2}\right) & v \geq 0,\ r \geq 0 \\ 0 & r < 0 \end{cases}$$

where $v^2$ is the power in the line of sight (LOS) component and $\mathrm{I}_0(\cdot)$ is the modified Bessel function of 0th order.

# Rician fading

The following is an illustration of Rayleigh and Rician distributions. We can see that due to the existence of a dominating term, Rician distribution is more concentrated at a finite positive value. Clearly, Rayleigh fading is the limiting-case situation of Rician distribution when $v$=0.



# 3.3 Overall fading effect

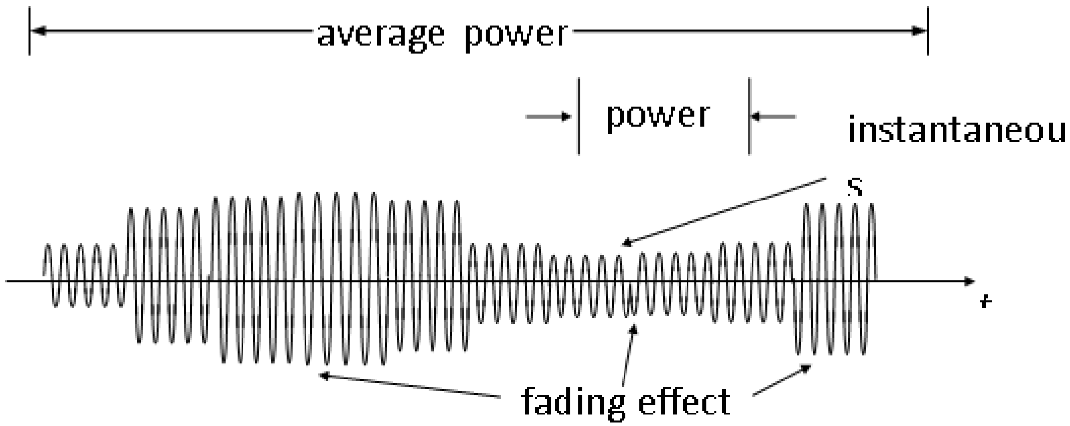
# Conventions on power and average power

We now consider the combined effects of large scale and short scale fading. We will use the following convention about power.

Instantaneous power = $|s(t)|^2$.

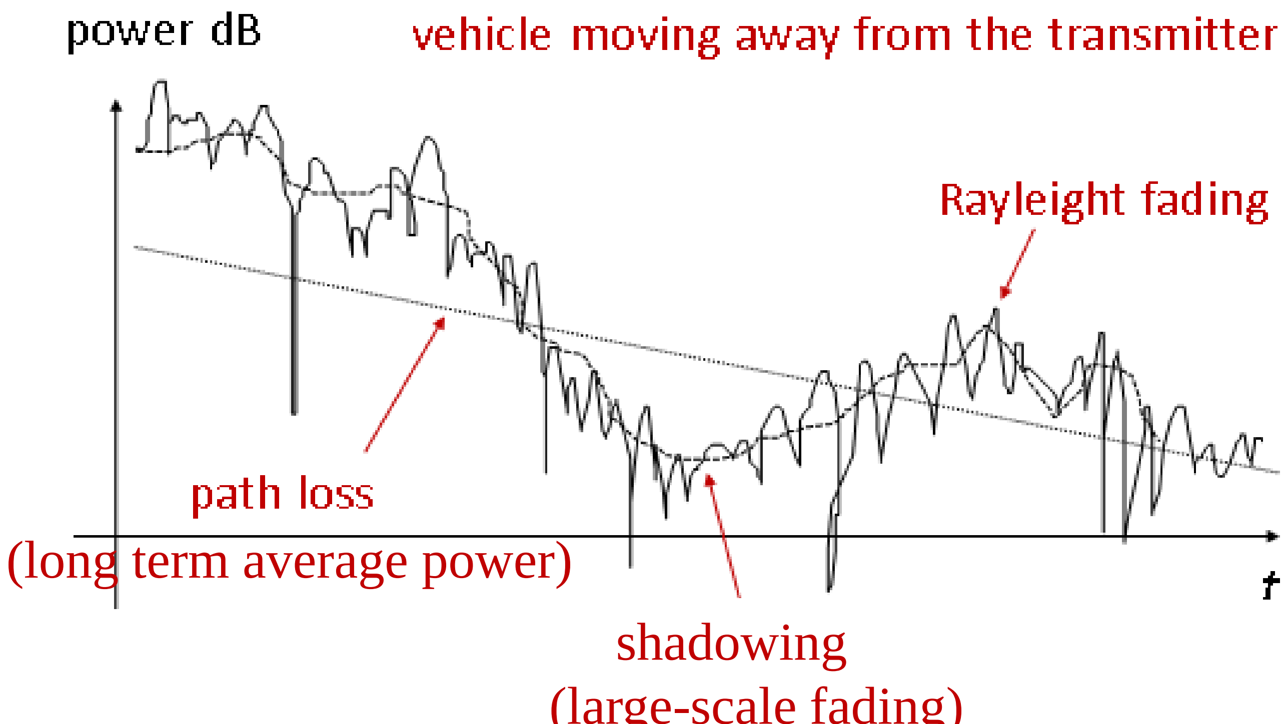
“Average power“ refers the local time mean of power. For example, the average power of a cosine function is 0.5. Notice that the average power is still a random variable, subject to fast or slow fading.

Long term average power = $E[|s(t)|^2]$ is averaged over a long term. Here a long term can be further divided into large-scale and small-scale fading cases.



# Overall channel effect

Usually we will not discuss instantaneous power. We will mostly discuss average power or long term average power.



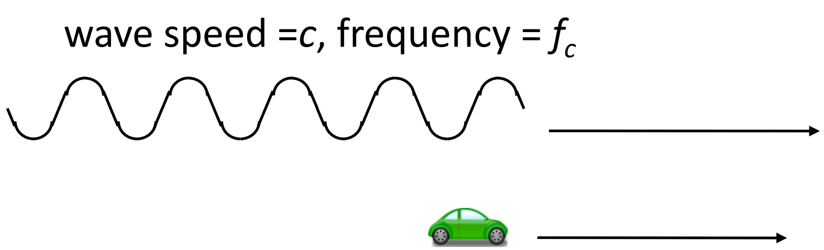
# 3.4 Doppler effect

# Doppler effect

We now discuss time varying channels related to, e.g., moving vehicles. We will focus on a narrowband transmitted signal $\cos(2\pi ft)$. When a vehicle is moving, its speed will affect the frequency of the received or the transmitted signal, which we call Doppler effect.

Consider an EM wave with carrier frequency $f_c$. Assume that a car is traveling in the same direction of the wave. If the speed of the car = 0, the wave received in the car apparently has frequency $f_c$. On the other hand, if the car is traveling at the speed of $v$=$c$ (speed of light), then the wave received in the car should have an apparent frequency = 0. (Why?) For any speed in between, the frequency of the wave relative to the car is given by,

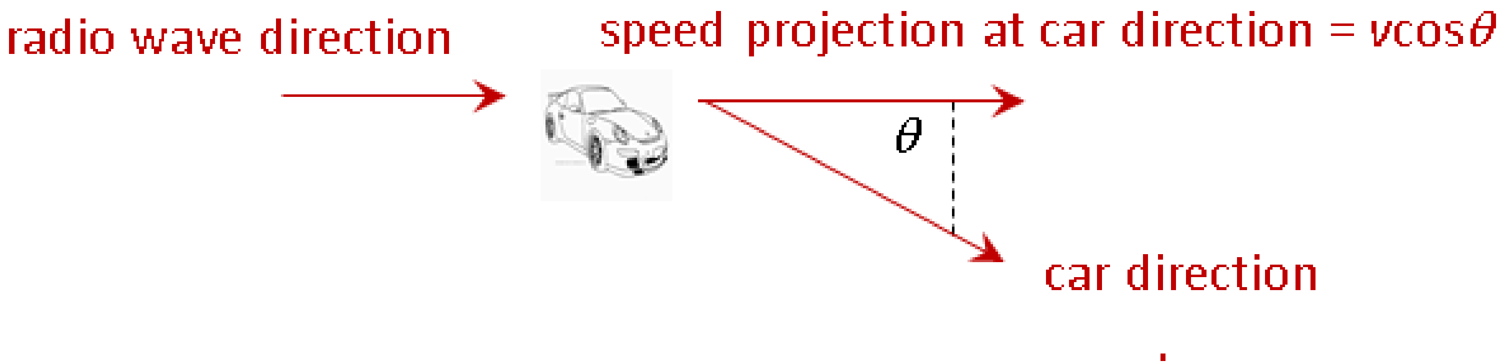
$$f = (1 - \frac{v}{c}) f_c$$



# Doppler effect

If the directions in which the car and the wave travel have an angle, then

$$f = \left(1 - \frac{v\cos\theta}{c}\right) f_c$$



# Doppler frequency

From the above, the Doppler effect introduces an distortion term in the frequency of the received signal measured by the Doppler shift

$$f_c - f = \Delta f = \frac{v f_c}{c} \cos\theta \qquad (12)$$

The maximum value of $\Delta f$ over all possible angle $\theta$ is called the Doppler frequency :

$$f_D = \frac{v f_c}{c} = \frac{v}{\lambda} \qquad (13)$$

The Doppler shift causes a carrier frequency shift in the received signal. For example, for a vehicle with $v$=72km/h=20m/s and $f$ =1000MHz ($\lambda$=0.3m), the maximum Doppler shift is $f_D$=20/0.3=66.7Hz.

# Doppler spectrum

With the Doppler effect, different rays of the received signal may have different frequencies, implying that $h_k$ will now change with time. Such changing is random but can be characterized by its statistical power spectrum. Aassume that the arrival angle of each ray is uniform distributed. The statistical power spectrum of the received signal is given by the so-called Doppler power spectrum below.
(See Jakes' model https://en.wikipedia.org/wiki/Rayleigh_fading.)

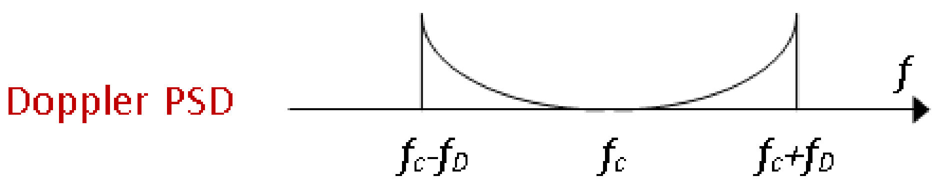
$$S_r(f) = \frac{P_r}{2\pi f_D} \frac{1}{\sqrt{1 - \left(\left|f - f_c\right| / f_D\right)^2}}$$

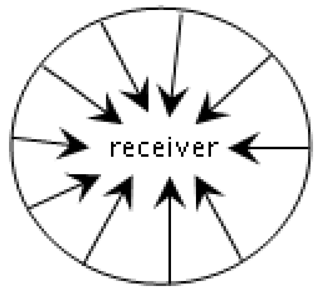
With normalized, we define a Doppler power spectral density (PSD) as

$$S(f) = S_r(f) / \int_{-\infty}^{\infty} S_r(f) df$$

# Physical meaning of Doppler spectrum

We may roughly interpreted a PSD as follows. We transmit a single-tone signal at frequency $f_c$. Then a PSD gives the probability density that the received signal for a particular path has frequency $f$. As seen from the PSD above, $f$ has high probability at $f_c \pm f_D$.



# 3.5 Time domain statistical modeling

# Channel coherent time

The channel coherent time, denoted by $T_c$, of a time-varying channel is the duration that a channel stays approximately unchanged. An approximate expression for $T_c$ is

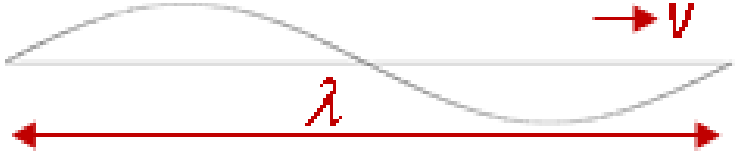
$$T_c \approx 1/f_D = \lambda/v$$

where $f_D$ is the Doppler frequency. Recall that $\lambda/v$ is the time to travel one wavelength at speed of light. Thus a channel stays approximately unchanged for a distance much smaller than one wavelength apart,.

More generally, the following is used

$$T_c = k\lambda/v$$

where $k$ is a constant. A typical value of $k$ is less than 1.



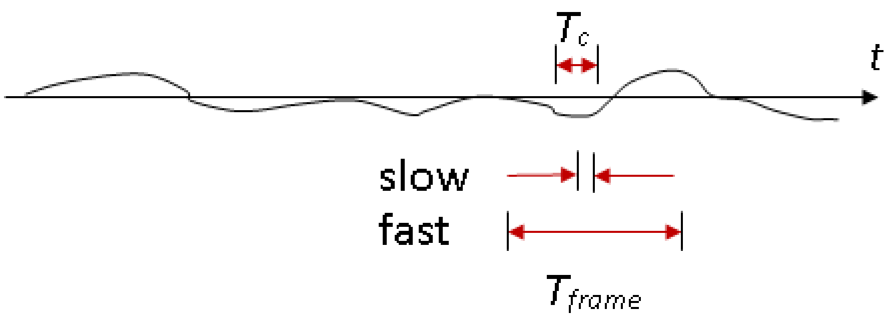
# Slow and fast fading

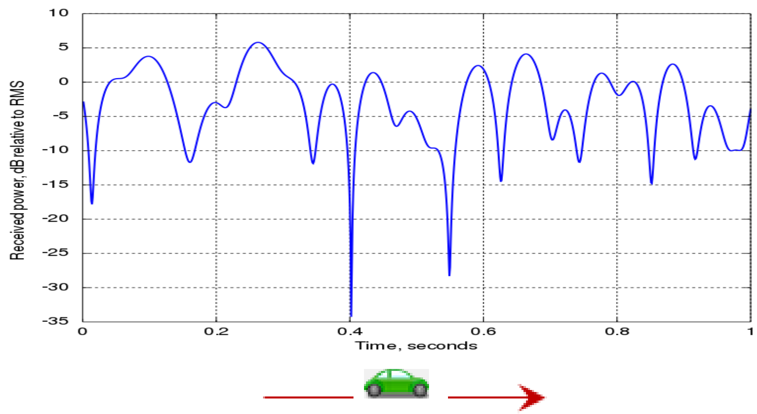
Let the length of a signal frame be $T_{frame}$. We say that a system is experiencing slow fading if

$$T_{frame} << T_c$$

On the other hand, we say that a system is experiencing fast fading if

$$T_{frame} >> T_c$$



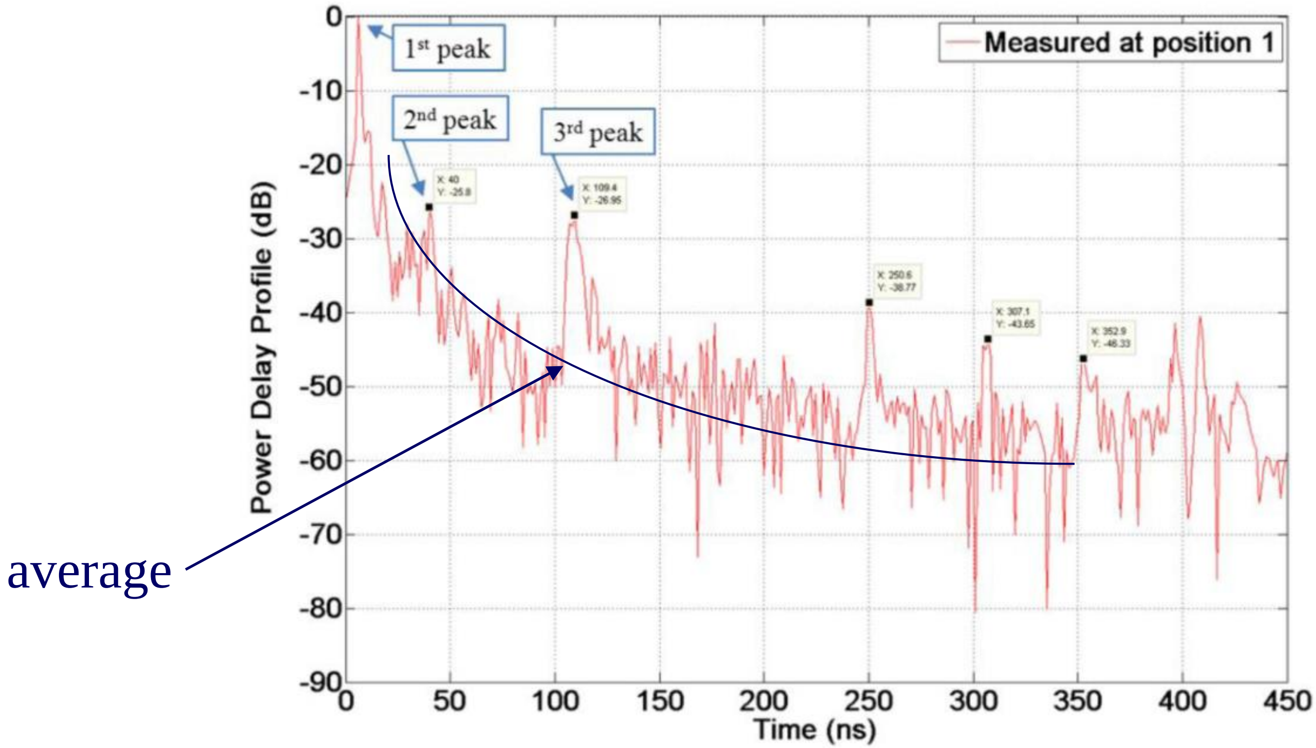




Note that "slow" or "fast" is relative ratio to $T_c=\lambda/v$.. It is not determined by $T_{frame}$ alone.

# 3.6 Delay spread

# Power delay profile

A power delay profile (PDP) gives the average received power through a multipath channel as a function of time delay $\tau$. The time delay is the difference in travel time between the transmitter and receiver. It is usually measured empirically and can be used to extract certain channel's parameters such as the delay spread.



# Power delay profile (PDP)

In a time varying channel the impulse response changes with both time and position, so it is a random function. A PDP, denoted by $f_{PDP}(\tau)$, is commonly used to characterize the power dispersion of signal in the time domain in a certain transmission environment. Roughly speaking, a PDP $f_{PDP}(\tau)$ gives the average received power at time $t+\tau$ for an impulse transmitted at time $t$. The expectation is over time and position.

The underlying assumption is that the channel is wide-sense stationary and ergodic, which roughly mean the following.

- The statistical behavior of a wide-sense stationary channel is unchanged over all time.
- The statistical behavior of an ergodic channel is unchanged over all positions and all tries.

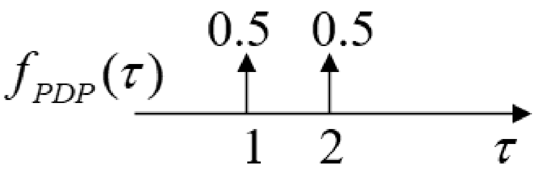
Keep in mind that the above are only assumptions.

# PDP after correlation receiver

A PDP after a correlation receiver becomes discrete. For example, consider

$$f_{PDP}(\tau) = 0.5 \cdot \delta(\tau - 1) + 0.5 \cdot \delta(\tau - 2)$$

This means that in average the received power is 0.5 at a delay $\tau$=1 and a delay $\tau$=2.




In average the received power is 0.5 at $\tau$=1 and 0.5 at $\tau$=2.

For simplicity, we may simply drop the impulse function and express a discrete PDP using a discrete function.

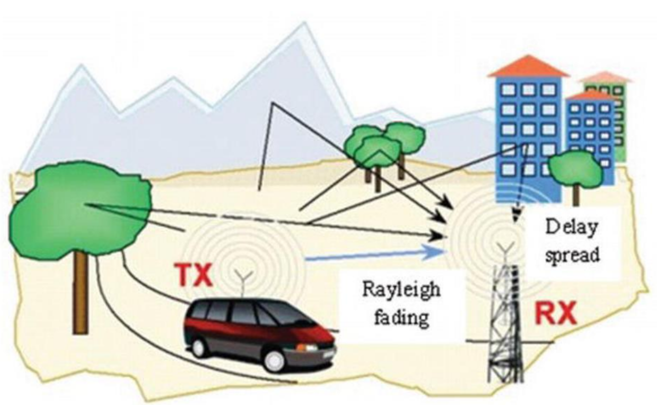
# Physical meaning of normalized PDP

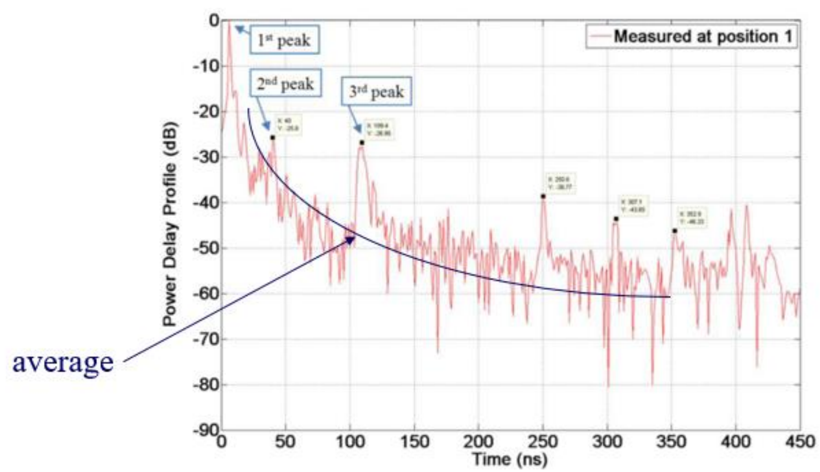
Given a $f_{PDF}(\tau)$, we defined a normalized PDP as:

$$f(\tau) = \frac{f_{PDP}(\tau)}{\int_0^{\infty} f_{PDP}(\tau) d\tau}$$

A normalized PDP can be seen as a distribution. Assume that we transmit an unit impulse signal at time $t$. We can interpret a normalized PDP $f(\tau)$ using the following two different views.

- The probability of a unit impulse received at time $t+\tau$ is given by $f(\tau)$.
- The average received power at time $t+\tau$ is given by $f(\tau)$.



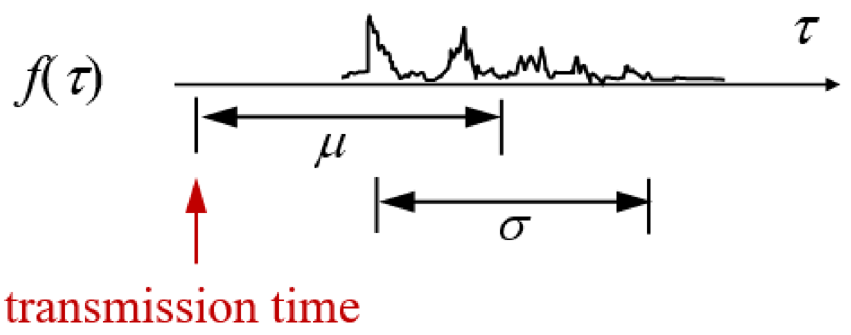



# Average delay and r.m.s. delay spreads

We define average delay and r.m.s. delay spread as

| | |
|---|---|
| average delay: | $\mu = \int_0^\infty \tau f(\tau) d\tau$ |
| r.m.s. delay spread: | $\sigma = \sqrt{\int_0^\infty (\tau - \mu)^2 f(\tau) d\tau}$ |

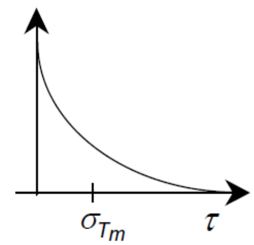
- $\mu$ is the average delay time, and
- $\sigma$ is the standard deviation of delay.



# Continuous and discrete exponential PDP

A common normalized PDP is given by an exponential distribution.

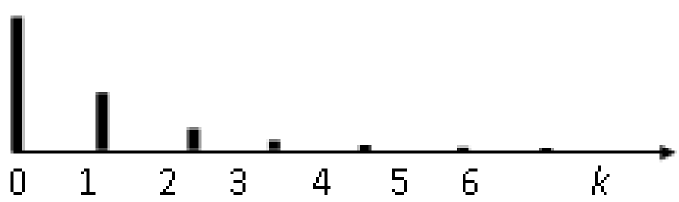
$$f(\tau) = \frac{1}{T} e^{-\frac{\tau}{T}}$$




It can be shown that $\mu = \sigma = T$ (i.e., mean = variance = $T$) for an exponential PDF. Other PDPs may not have such a property.

We can also define a normalized discrete exponential PDP.

$$f(k) = \left(1 - e^{-\frac{1}{T}}\right) e^{-\frac{k}{T}}$$



# 3.7 Frequency domain statistical modeling

# Channel coherent bandwidth

Similar to the channel coherent time defined earlier, the coherent bandwidth, denoted by $B_c$, is the frequency range that channel can be approximately regarded as unchanged. A commonly used rule is

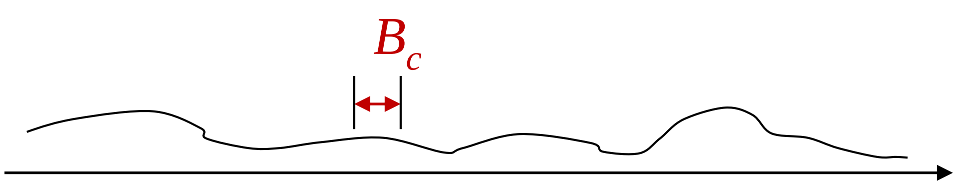
$$B_c \approx 1/\sigma_{Tm}$$

where $\sigma_{Tm}$ is the rms delay spread.

More generally, the following is used

$$B_c = k/\sigma_{Tm}.$$

where $k$ is a constant. A typical value of $k$ is less than 1



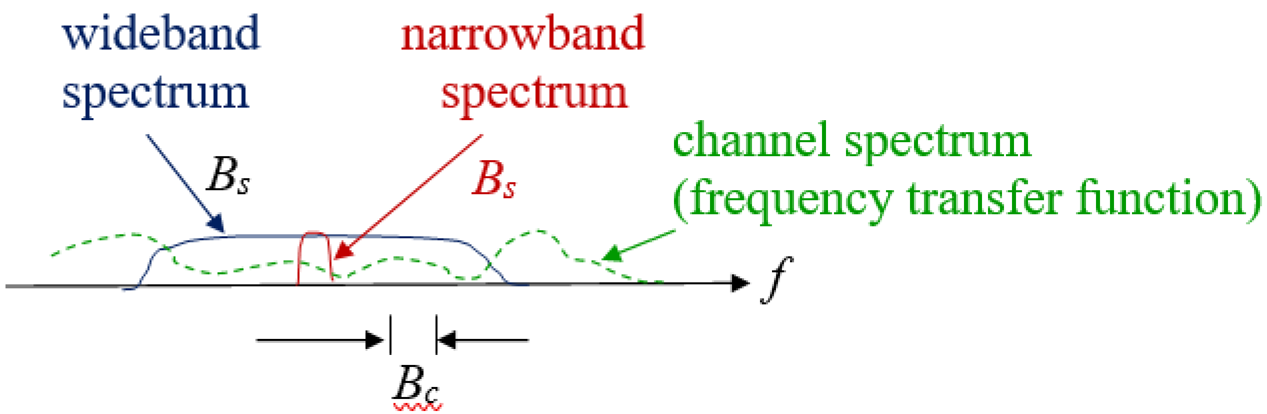

frequency transfer function of the channel

# Narrowband and wideband systems

Let the bandwidth of the signal be $B_s$. We say that the system is narrowband if

$$B_s << B_c .$$

Otherwise, we say that the system is wideband.




Note that "wideband" or "narrowband" is relative to the channel coherent bandwidth. It is not determined by signal bandwidth alone.

# Flat and frequency-selective fading

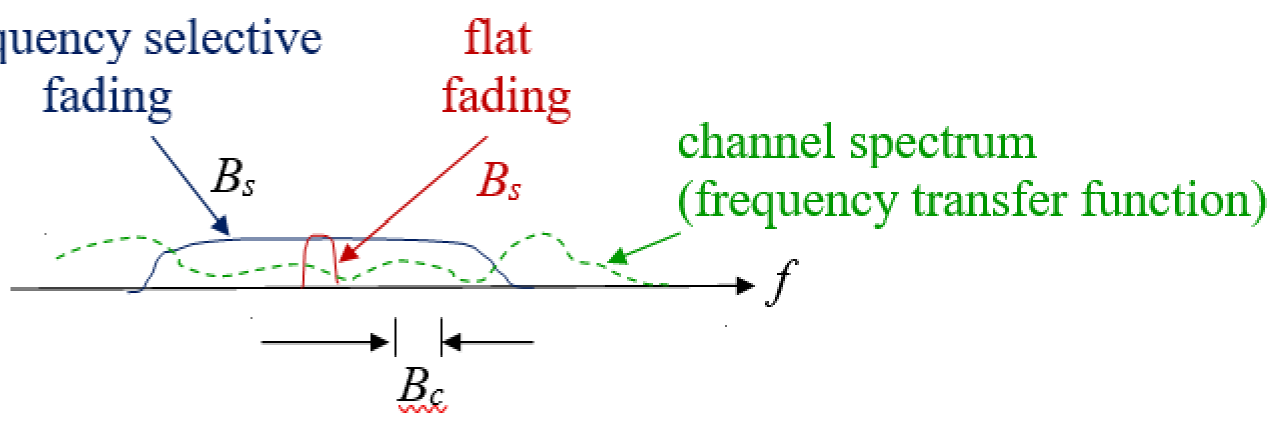
Also, we may say that fading is flat if

$$B_s << B_c \ .$$

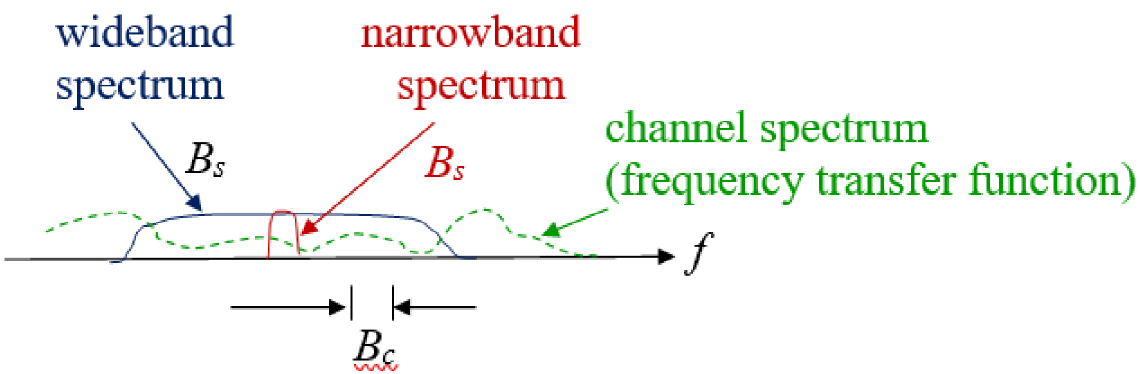
Otherwise, we say that fading is frequency-selective.

Clearly, a narrow band system is experiencing flat fading. In this case, the received power spectrum is roughly flat.

A wide band system is experiencing frequency-selective fading. In this case, the received power spectrum varies at different frequency range.



# Observations




"Fast" or "slow" is determined by both frame length and Doppler spread. (Recall that $T_c \approx 1/f_D$.)

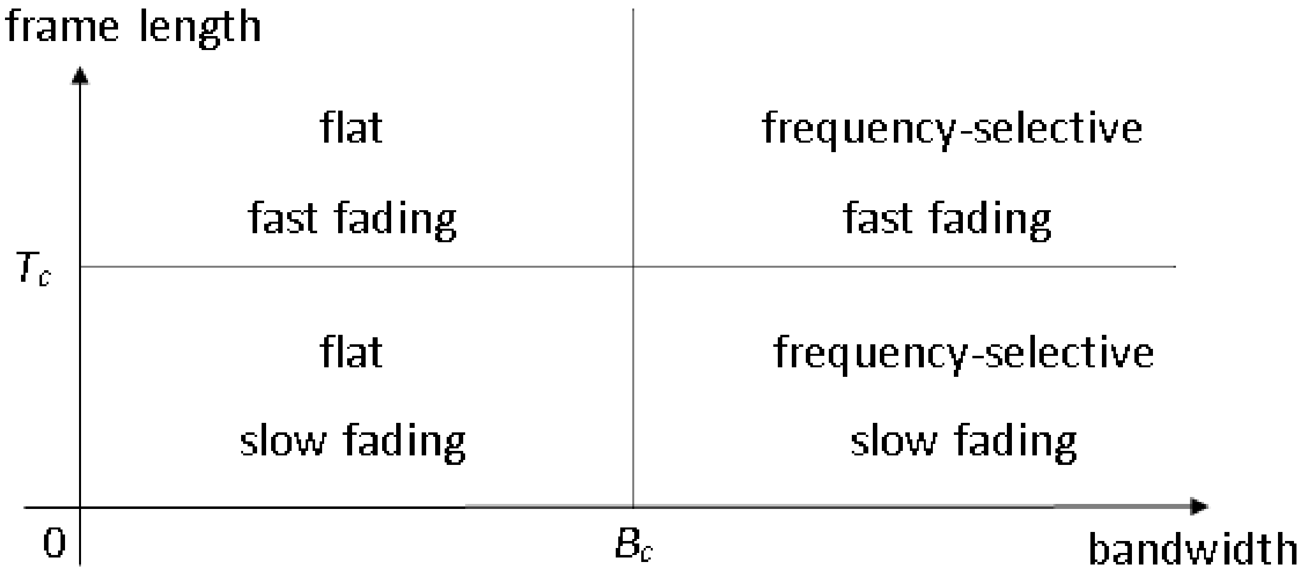
"Flat" and "frequency selective" are determined by both system bandwidth and delay spread. (Recall that $B_c \approx 1/\sigma_{Tm}$.)

It is interesting to observe the following.

- The coherence in the time domain is determined by a frequency parameter Doppler frequency.
- The coherence in the frequency domain is determined by a time parameter delay spread.

# Summary on different types of fadings

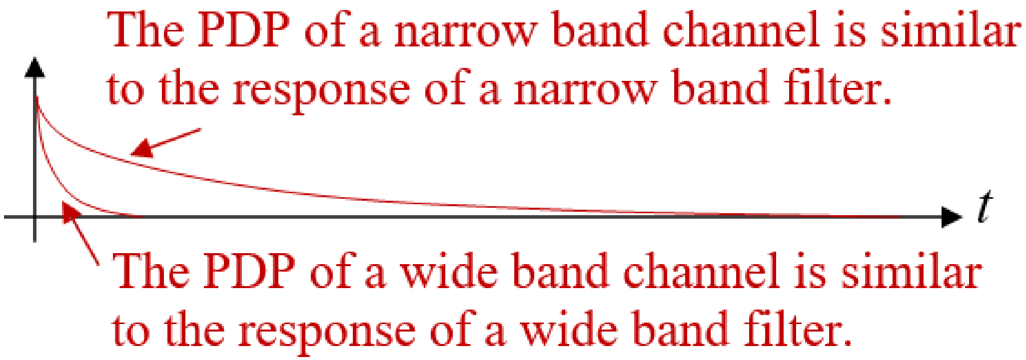
The following figure summarizes different types of fading channels.




Notes: The terms "slow" or "fast", however, are sometimes used with other meanings. Slow fading may mean "large-scale" fading including shadow fading and path-loss caused by building or large objects. Fast fading may mean small-scale fading, i.e., Rayleigh fading, caused by multiple reflections. This is somewhat confusing and you should be careful about the usage

# Connection of PDP and coherent bandwidth

We can treat the channel as a filter. If an impulse response decays slowly, than the bandwidth a filter is a relatively narrow. Otherwise, the bandwidth is relatively wide.



# 3.8 Channel simulation

# Channel simulation

The field experiment of a wireless system is costly. Computer simulation is a more efficient way for assessing system performance. Generating channel coefficients is a key step in simulation.

$$h_k = \sum_m h_{k,m}.$$

Combining the discussions above, we model the channel coefficients of a multi-path, time-varying channel with Doppler effect as follows. We assume the channel coefficients are functions of $t$ and write them as $\{h_{k,m}(t)\}$. Each $h_{k,m}(t)$ corresponds to a transmitted signal at time $t$ and can be expressed as:

$$h_{k,m}(t) = A\alpha_{k,m} e^{-j2\pi\left(\Delta f_{k,m} t + \phi_{k,m}\right)}$$

Here $\alpha_{k,m}$, $\Delta\phi_{k,m}$ and $\phi_{k,m}$ represent, respectively, amplitude, frequency shift and initial phase caused by a reflection.

# Channel simulation

The variables $\alpha_{k,m}$, $\Delta\phi_{k,m}$ and $\phi_{k,m}$ mentioned in the previous page are all random variables. They can be modelled as follows.

- Randomly draw $\alpha_{k,m}$ using a discrete PDP (such as an exponential distribution).
- Randomly draw $\Delta f_{k,m}$ based on a Doppler PDF determined by vehicle speed.
- Randomly draw $\phi_{k,m}$ using a uniform distribution.

Note that $m$ is an index of random samples. If a sufficient number of samples are taken, the summation in (6) results in Rayleigh fading.

# Chapter 3 Summary

1) Equivalent discrete channel model

$$y_n = \sum_k h_k x_{n-k} + \eta_n \, .$$

2) Rayleigh and Rician distributions. In particular, the Rayleigh distribution is define by

$$p_r(r) = \begin{cases} \dfrac{r}{\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r \geq 0 \end{cases}$$

An exponential distribution is defined by

$$p_z(z) = \begin{cases} \dfrac{1}{2\sigma^2} e^{-\frac{z}{2\sigma^2}} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

3) The Doppler frequency is given by

$$f_D = \frac{v}{c} f_c$$

The related Doppler PSD is given by

$$p_f(f) = \frac{P_r}{2\pi f_D} \frac{1}{\sqrt{1 - (|f - f_c| / f_D)^2}}$$

# Chapter 3 Summary

4) Continuous PDP: $$f_{PDP}(\tau) = (1/T)e^{-\tau/T}$$

Discrete PDP: $$f_{PDP}(k) = (1 - e^{-1/T})e^{-k/T}$$

5) Average delay: $$\mu = \int_0^\infty \tau f(\tau) d\tau$$

r.m.s. delay spread: $$\sigma = \sqrt{\int_0^\infty (\tau - \mu)^2 f(\tau) d\tau}$$

(6) Channel coherent time: $$T_c \approx 1/f_D = \lambda / v$$

Channel coherent bandwidth: $$B_c \approx 1/\sigma_{Tm}$$

7) Different fading types:

- flat fast fading
- frequency-selective fast fading
- flat slow fading
- frequency-selective slow fading